McGILL UNIVERSITY

FACULTY OF SCIENCE

IN-DEPARTMENT FINAL EXAMINATION

MATH 574

Dynamical Systems

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Date: Wednesday April 16, 2008
Time: 2:00PM - 5:00PM

INSTRUCTIONS

Please attempt all questions in the exam booklets provided.
All questions carry equal weight.
This is a closed book exam. No notes, and text books are permitted .
Non-programmable and non-graphical calculators are permitted.
Dictionaries are not permitted.

This exam of comprises the cover page, and 3 pages of 5 questions.

1. Consider Odell's predator-prey model

$$\dot{x} = x[x(1-x) - y],$$
$$\dot{y} = y(x - \mu),$$

where $x \geqslant 0$ is the dimensionless population of the prey, $y \geqslant 0$ is the dimensionless population of the predator and μ is a positive parameter.

(a) Find all the fixed points of this dynamical system and determine their linear stability types.

(b) For $\mu > 1$, sketch a phase portrait for the dynamical system. Include the isoclines (nullclines) on your phase portrait. State, without proof, $\omega(x_0, y_0)$ for all $x_0 \geqslant 0$, $y_0 \geqslant 0$. What is the ecological significance?

(c) Find a potential Hopf bifurcation point for the non-trivial fixed point, and show that $Re(\frac{d\lambda}{d\mu}) \neq 0$ at this point (but do *not* check the $a \neq 0$ degeneracy condition). Near the bifurcation point, what will be the approximate period of any periodic orbits created in the bifurcation?

(d) What does it mean for a Hopf bifurcation to be supercritical or subcritical? Given that the Hopf bifurcation occurs and is supercritical in Odell's model, sketch two plausible phase portraits for μ close to Hopf bifurcation point, one for μ each side of the bifurcation. What is the ecological significance of these phase portraits?

2. Define the linear stable, linear centre and linear unstable manifolds for a non-hyperbolic fixed point. State the stable manifold theorem.

Consider the fixed point at $(x, y, z) = (0, 0, 0)$ of the Lorenz equations with $r = 1$:

$$\dot{x} = \sigma(y - x),$$
$$\dot{y} = x - y - xz,$$
$$\dot{z} = xy - bz.$$

(a) Determine the linear stable, centre and unstable manifolds of the fixed point.

(b) Assuming that the centre manifold may be written as

$$y = g(x) = \sum_{i=1}^{\infty} c_i x^i, \qquad z = h(x) = \sum_{i=1}^{\infty} k_i x^i,$$

find c_i, k_i for $i = 1, 2, 3$ and thus find α_i for $i = 1, 2, 3$ such that the dynamics on the centre manifold is given by

$$\dot{x} = \sum_{i=1}^{\infty} \alpha_i x^i.$$

Is the fixed point $(x, y, z) = (0, 0, 0)$ of the Lorenz equations asymptotically stable?

The following two questions concern the system of differential equations $\dot{u} = f(u)$ *where* $u(t) = (x(t), y(t), z(t)) \in \mathbb{R}^3$ *and*

$$\begin{aligned} \dot{x} &= \sigma(y - x) - yz, \\ \dot{y} &= rx - y, \\ \dot{z} &= xy - 2z, \end{aligned} \tag{1}$$

3. Consider the system of ODEs (1) where r and σ are positive constants.

   (a) Let $V(x, y, z) = x^2 + y^2 + (z - r - \sigma)^2$. Evaluating V along a solution trajectory and considering it as a function of time show that

   $$\frac{1}{2}\frac{d}{dt}V(t) = -\sigma x^2 - y^2 - 2z^2 + 2(r + \sigma)z \leqslant -\sigma x^2 - y^2 - (z - r - \sigma)^2 + (r + \sigma)^2.$$

   Hence, or otherwise, show that the system of differential equations defines a dynamical system for any $r > 0$, $\sigma > 0$.

   (b) By showing that $W(x, y, z) = x^2 + \sigma y^2 + z^2$ is a Lyapunov functional, or otherwise, show that the fixed point $(x, y, z) = (0, 0, 0)$ is globally asymptotically stable for $r \in (0, 1)$.

   (c) Show that the divergence of the vector field (sometimes written $\nabla.f$) is always negative. What (briefly) is the significance of this?

4. Consider the system of ODEs (1) where r and σ are positive constants.

   (a) What does it mean for a fixed point of (1) to be hyperbolic? Show that the Jacobian matrix at the fixed point $(x, y, z) = (0, 0, 0)$ always has real eigenvalues, and determine for which parameter values it is *not* hyperbolic. For parameter values for which it is hyperbolic, determine the stability of the fixed point.

   (b) Solve directly for all the fixed points of the dynamical system as functions of the parameters. What type of bifurcation is observed at the change of stability of the fixed point $(x, y, z) = (0, 0, 0)$?

   (c) Show that the characteristic equation of the Jacobian matrix at the nonzero fixed points is

   $$(\lambda + 2)[\lambda^2 + (1 + \sigma)\lambda + 2\sigma(r - 1)] = 0,$$

   and hence show that these fixed points are always stable, and that the dynamical system has no other local (fixed point or Hopf) bifurcations.

   (d) When $\sigma = 9$ and $r = 4$ which eigenvalue is associated with the so-called "dominant" decay direction for the non-zero fixed points. Briefly, (without computing any eigenvectors), sketch the unstable manifold of the fixed point at $(x, y, z) = (0, 0, 0)$ in this case. Include all the fixed points in your sketch.

5. Consider the tent-map $f : [0,1] \to [0,1]$ defined by

$$f(x) = \begin{cases} 2x & x \in [0, 1/2] \\ 2(1-x) & x \in [1/2, 1] \end{cases}$$

and the discrete dynamical system it defines via the iteration $x_{n+1} = f(x_n)$.

(a) Find the eventually fixed points which satisfy $f(x) = 0$ or $f^2(x) = 0$ or $f^3(x) = 0$ or $f^n(x) = 0$ for some $n > 1$, and hence show that eventually fixed points are dense in $[0, 1]$.

(b) How many fixed points, period-two points, period 3 points and period n points does f have? Show that periodic points are dense in $[0, 1]$.

(c) How many periodic points does f have with prime period 3, and how many distinct orbits are there of prime period 3? State Sharkovskii's theorem. What does it imply for this dynamical system?

(d) Let x be a point of prime period n for arbitrary n. Show that any such periodic orbit is unstable. Show that the preimages of x (that is $\{y \in [0, 1] : f^m(y) = x \text{ for some } m > 0\}$) are dense in $[0, 1]$.