MATH 550. Winter 2012.

Final Exam.

Due by 5PM on Monday, April 30th in Burnside 1114.

1. **LYM inequality for set pairs.**

   (a) Let $A_1, A_2, \ldots, A_m, B_1, B_2, \ldots, B_m$ be finite sets such that $A_i \cap B_j = \emptyset$ if and only if $i = j$. Show that
   $$\sum_{i=1}^{m} \frac{1}{\binom{|A_i|+|B_i|}{|A_i|}} \leq 1.$$

   (b) Show that (a) implies LYM inequality.

2. **Turán density.**

   Let H denote the 3-uniform hypergraph on 6 vertices and 8 edges with
   $$V(H) = \{a_1, a_2, b_1, b_2, c_1, c_2\}$$
   and
   $$E(H) = \{\{a_i, b_j, c_k\} \mid i, j, k \in \{1, 2\}\}.$$
   Show that $\pi(H) = 0$. (For every $\epsilon > 0$ there exists n_0 such that if G is a 3-uniform hypergraph on $n \geq n_0$ vertices, containing no copy of H then G has at most $\epsilon\binom{n}{3}$ edges.)

3. **Application of Fractional Helly Theorem.**

   (a) Prove that if a collection of n convex sets in $\mathbb{R}^2$ has the property that out of every 4 sets some three have a point in common then there is a point that belongs to at least $n/12$ sets in the collection.

   (b) Prove that for all positive integers p, d so that $p \geq d + 1$ there exists a constant $c = c(d, p) > 0$ so that if a family of $n \geq p$ convex sets in $\mathbb{R}^d$ has the property that among any p sets some $d + 1$ have a point in common then some point belongs to at least cn sets in the family.

   (c) Prove that for every positive integer d there is a constant $c = c(d)$ such that if a family $\mathcal{F}$ of n convex sets in $\mathbb{R}^d$ has the property that out of any $d + 2$ sets in $\mathcal{F}$ some $d + 1$ have a point in common, then $\mathcal{F}$ can be partitioned into at most $c \log n$ intersecting sub-families.

4. **Extending Erdős-Szekeres theorem.**

   Let $f : \mathbb{N} \times \mathbb{N} \to \mathbb{R}$ be a function. We say that a sequence of natural numbers $a_1 < a_2 < \ldots < a_k$ is *f-convex* if $f(a_1, a_2) \leq f(a_2, a_3) \leq \ldots \leq f(a_{k-1}, a_k)$. We say that it is *$f$-concave* if $f(a_1, a_2) \geq f(a_2, a_3) \geq \ldots \geq f(a_{k-1}, a_k)$. Let $r_f(k, l)$ denote the minimum integer N such that every set of N natural numbers contains an f-convex subsequence of length k or an f-concave subsequence of length l.

   (a) Show that $r_f(3, l) \leq l$ and $r_f(k, 3) \leq k$ for all $k, l \geq 3$ and all functions f.

   (b) Show that $r_f(k, l) \leq r_f(k-1, l) + r_f(k, l-1) - 1$ for all $k, l \geq 4$.

   (c) Deduce that
   $$r_f(k, l) \leq \binom{k+l-4}{k-2} + 1$$
   for all $k, l \geq 3$.

   (d) Suppose that $f(m, n) = g(n)$ for some function $g : \mathbb{N} \to \mathbb{R}$. Show that $r_f(k, l) \leq r_f(k-1, l) + l - 2$ for $l \geq 3$. Deduce that $r_f(l, l) \leq l^2 - 4l + 6$ for $l \geq 3$.

5. **Combinatorial Nullstellensatz.**

   Let G be a graph containing a Hamiltonian cycle. Suppose that every vertex $v \in V(G)$ is assigned a set $S(v)$ of two distinct real numbers. Show that it is possible to choose a number $c(v) \in S(v)$ for every vertex $v \in V(G)$, so that $\sum_{w \in N(v)} c(w) \neq 0$ for every $v \in V(G)$.

   (A *Hamiltonian cycle* is a cycle containing every vertex of the graph. We denote by $N(v)$ the set of all vertices adjacent to the vertex v.)

6. **Shannon capacity of the seven cycle.**

   Let $\Theta(C_7)$ denote Shannon capacity of the cycle of length 7. Show that
   $$\sqrt{10} \leq \Theta(C_7) \leq \frac{7}{2}.$$