McGILL UNIVERSITY

FACULTY OF SCIENCE

FINAL EXAMINATION

MATH 447

STOCHASTIC PROCESSES

Examiner: Professor W. Anderson
Associate Examiner: Professor M. Asgharian

Date: Wednesday April 19 , 2005
Time: 2:00 p.m - 5:00 p.m

INSTRUCTIONS

Please answer **FIVE** complete questions in the exam booklets provided.
This is a closed book exam.
No aids other than pocket calculators are allowed
Use of a regular and/or translation dictionary is permitted

This exam comprises the cover page, and two [redacted] page of 6 questions.

(1) (a) Let X and Y be discrete random variables. Show that

$$E(XY) = E[YE(X|Y)].$$

(b) A prisoner is trapped in a cell containing three doors. The first door leads to a tunnel which returns him to his cell after three day's travel. The second leads to a tunnel which returns him to his cell after four day's travel. The third door leads to freedom after one day's travel. Let T denote the number of days until the prisoner reaches freedom.

Assuming that the prisoner will always select doors 1,2 and 3 with probabilities .5, .3, and .2, find $E(T)$.

(2) (a) Let $\{Z_n, n \geq 0\}$ be a Galton-Watson branching process with $Z_0 = 1$. Let $P_n(s)$ be the probability generating function of Z_n.

i. Show that $P_{n+1}(s) = P_n(P_1(s)) = P_1(P_n(s))$ for $n \geq 0$.

ii. Show that the probability of extinction of the process is a root of the equation $P_1(s) = s$.

(b) A certain species of marine algae is found to reproduce every hour as follows. Each organism produces k new organisms of the same type with probability p_k, where $p_0 = .25$, $p_1 = .25$, and $p_2 = .50$; and then dies. An hereditary colour mutation is detected in four of these organisms at time $n = 0$. Calculate the average number of mutant organisms alive at any time n, and the probability of extinction of the mutation.

(3) A Markov chain $\{X_n, n \geq 0\}$ has state space $E = \{1, 2, 3, 4, 5\}$ and transition matrix

$$P = \begin{pmatrix} \frac{3}{4} & 0 & 0 & \frac{1}{4} & 0 \\ \frac{1}{5} & \frac{2}{5} & \frac{2}{5} & 0 & 0 \\ 0 & 0 & \frac{3}{4} & 0 & \frac{1}{4} \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{5}{8} & 0 & \frac{3}{8} \end{pmatrix}.$$

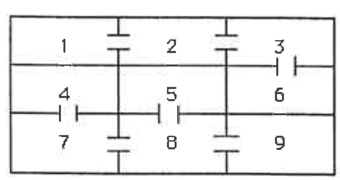
(a) Find $\Pr\{X_5 = 1 | X_3 = 4\}$.

(b) Find all communicating classes and classify all states.

(c) If the chain starts out in state 1, what is the expected number of transitions until it returns to state 1?

(4) (a) Prove that in a finite Markov chain, at least one of the states must be recurrent.

(b) A white rat is put into the maze shown below at time 0. At each instant $n = 1, 2, \ldots$, it moves to a new compartment, or stays in the same compartment, each with probability 1/2. If it moves, and there are k ways to leave the compartment, it chooses one of these at random. If the rat starts out in compartment 7, find the expected time at which it returns to compartment 7.



(5) (a) Show how to derive the Kolmogorov equations for a birth and death process $\{X(t), t \geq 0\}$ with birth coefficients λ_n and death coefficients μ_n.

(b) Suppose that $\{X_t, t \geq 0\}$ is a birth and death process with $X_0 = i$, birth parameters $\lambda_n = n\lambda$, $n \geq 0$, and death parameters $\mu_n = n\mu, n \geq 1$. Show that the average population size $E(X_t)$ at time t is

$$E(X_t) = ie^{(\lambda - \mu)t}.$$

(6) (a) Briefly define the following terms.

i. $G/G/m$ queueing system
ii. $M/M/1$ queueing system
iii. traffic intensity

(b) The $M/M/1/K$ Finite Storage Queue is the same as the $M/M/1$ queueing system except that the system has a maximum capacity (including the customer being served) of K customers. Show that in this case, the steady state probabilities $p_k = \Pr\{k \text{ customers in system as } t \to \infty\}$ are

$$p_k = \begin{cases} \frac{1-(\lambda/\mu)}{1-(\lambda/\mu)^{K+1}} \left(\frac{\lambda}{\mu}\right)^k & \text{if } 0 \le k \le K, \\ 0 & \text{otherwise.} \end{cases}$$