FACULTY OF ENGINEERING

FINAL EXAMINATION

MATHEMATICS MATH381

Complex Variables and Transforms

Examiner: Professor S. W. Drury

Date: Monday, 17 December 2007

Associate Examiner: Professor N. Sancho

Time: 9: 00 am. – 12: 00 noon.

INSTRUCTIONS

Answer all questions.
This is a closed book examination.
Calculators are not permitted.

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This exam has 7 questions and 3 pages

1. (i) (5 points) Let $u(x,y) = 3e^x \sin(y) + 4xy$. Find the real function $(x,y) \mapsto v(x,y)$ such that $u + iv$ is analytic and $v(1,0) = 2$.

(ii) (5 points) Using parametrization, find the integral $\int_\Gamma \bar{z}^2 dz$ where Γ is the circle $|z-1| = 2$ traversed in the anticlockwise sense.

2. (i) (5 points) Use the theory of residues to evaluate $\int_\Gamma \frac{\text{Log}(z)}{z^2(z-1)^3} dz$ where Γ is the circle $|z-2| = \frac{3}{2}$ traversed in the anticlockwise sense and Log denotes the principal branch of the logarithm.

(ii) (5 points) The function $f(z) = \frac{z}{(1+z^2)(2-z^2)\sin(z)}$ is expanded as a series $\sum_{n=0}^{\infty} c_n(z-\alpha)^n$ where $\alpha = \frac{1}{2} + \frac{1}{3}i$. What will the radius of convergence be?

3. A function f has a Laurent expansion $f(z) = \sum_{n=-\infty}^{\infty} a_n z^n$ where

$$a_n = \begin{cases} \frac{1}{n!} & \text{if } n \geq 0, \\ 3 & \text{if } n = -1, \\ 1 & \text{if } n \leq -2. \end{cases}$$

(i) (3 points) Where is the Laurent expansion valid?
(ii) (3 points) Find a formula for the function f.
(iii) (4 points) Another Laurent expansion $\sum_{n=-\infty}^{\infty} b_n(z-1)^n$ valid in a punctured disc $0 < |z-1| < r$ with $r > 0$ suitably chosen, agrees with $f(z)$ for z in a nonempty domain. Find the coefficients b_n explicitly.

4. (i) (3 points) If $F(z) = z^{-4}(z^3+1)$, find the Inverse $\mathbb{Z}$–Transform, $\mathbb{Z}^{-1}[F(z)] = f(nT)$, $n \geq 0$.

(ii) (3 points) If $G(z) = \dfrac{z}{(z-1)^2}$, find the Inverse $\mathbb{Z}$–Transform, $\mathbb{Z}^{-1}[G(z)] = f(nT)$, $n \geq 0$.

(iii) (4 points) If $H(z) = (z^2-1)^{-\frac{1}{2}}$ is the branch which is positive for $z > 1$ and $\mathbb{Z}^{-1}[H(z)] = h(nT)$, find the Inverse $\mathbb{Z}$–Transform $h(5T)$.

5. (10 points) Use residue calculus to evaluate $\displaystyle\int_{-\infty}^{\infty} \frac{x^4}{x^6+1} dx$.

6. For a function f defined on the real line, the Fourier transform F is given by

$$F(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(t) e^{-i\omega t} dt.$$

(i) (5 points) If $F(\omega) = \dfrac{1}{\omega^2 + a^2}$, find f being careful to distinguish between $f(t)$ for $t > 0$ and $t < 0$.

(ii) (5 points) Find the inverse Fourier transform of the function

$$G(\omega) = \frac{\omega^2+4}{(\omega^2+1)(\omega^2+9)}.$$

7. (i) (4 points) Write down the Bromwich integral which gives the Inverse Laplace Transform $f(t)$ of

$$F(s) = \frac{s}{(s^2+2s+2)(s-1)^2},$$

including a diagram of the path and the singularities.

(ii) (6 points) Evaluate $f(t)$ for $t > 0$. Your answer should consist of real functions only.

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