FACULTY OF SCIENCE

FINAL EXAMINATION

MATHEMATICS MATH 355

Analysis 4

Examiner: Professor S. W. Drury

Date: Wednesday, April 18, 2007

Associate Examiner: Professor K. N. GowriSankaran

Time: 2: 00 pm. – 5: 00 pm.

INSTRUCTIONS

Attempt six questions for full credit.

This is a closed book examination.
Write your answers in the booklets provided.
All questions are of equal weight, each is alloted 20 marks.

This exam has 7 questions and 4 pages

1. (i) (4 points) Define the concepts field and σ-field.
   (ii) (2 points) Define the concept of premeasure on a field and measure on a σ-field.
   (iii) (2 points) Define the concept of outer measure.
   (iv) (4 points) State the Carathéodory Extension Theorem.
   (v) (8 points) If μ is a premeasure on a field $\mathcal{F}$ of subsets of X and $\mu^\star$ is the outer measure it defines on X by the equation $\mu^\star(A) = \inf \sum_{j=1}^{\infty} \mu(A_j)$ where the infimum is taken over all possible sequences of sets $A_j \in \mathcal{F}$ such that $A \subseteq \bigcup_{j=1}^{\infty} A_j$, show that for any subsets A and B of X that $\mu^\star(A \cup B) + \mu^\star(A \cap B) \leq \mu^\star(A) + \mu^\star(B)$.

2. Let $(X, \mathcal{M}, \mu)$ be a measure space.
   (i) (5 points) Under what conditions can one define $\int f(x)d\mu(x)$ for a signed $\mathcal{M}$-measurable function f on X? In this case give the definition in terms of the integral of nonnegative $\mathcal{M}$-measurable functions on X.

   Let g be a nonnegative $\mathcal{M}$-measurable function on X satisfying $\int g(x)d\mu(x) < \infty$.

   (ii) (5 points) Prove Tchebychev's inequality $\mu(\{x; g(x) > t\}) \leq \frac{1}{t} \int g(x)d\mu(x)$ for $t > 0$.

   (iii) (10 points) Let $\mu(X) = 1$ and let f be a signed $\mathcal{M}$-measurable function such that $\int f d\mu = 0$ and $\int f^2 d\mu = 1$. Show that $\mu(\{x; f(x) > s\}) \leq \frac{1}{1+s^2}$ for $s > 0$.

   Hint: Consider $g(x) = (sf(x) + 1)^2$.

3. (i) (5 points) State the Monotone Convergence Theorem.
   (ii) (5 points) State the Dominated Convergence Theorem.
   (iii) (10 points) Find $\lim_{n\to\infty} n \int_0^{\infty} \frac{1}{1+x^4} \sin\left(\frac{x}{n}\right) dx$. In answering the question you may use the inequality $|\sin(u)| \leq \min(1, |u|)$. Otherwise, justify all steps and for full credit simplify your answer as much as possible.

4. (i) (5 points) Let $(X, \mathcal{S})$ and $(Y, \mathcal{T})$ be measurable spaces. Define $\mathcal{S} \otimes \mathcal{T}$.

   If X is a metric space, we denote $\mathcal{B}_X$, its Borel σ-field.

   (ii) (15 points) Prove in detail that $\mathcal{B}_{\mathbb{R}} \otimes \mathcal{B}_{\mathbb{R}} = \mathcal{B}_{\mathbb{R}^2}$.

5. (i) (5 points) State Tonelli's Theorem.
(ii) (5 points) State Fubini's Theorem.
(iii) (10 points) Starting from the identity

$$\int_0^\infty e^{-sx}\sin(ux)dx = \frac{u}{u^2+s^2}$$

valid for $s > 0$ and $u \in \mathbb{R}$, show that

$$\int_0^\infty e^{-sx}\frac{1-\cos(tx)}{x}dx = \tfrac{1}{2}\ln(s^2+t^2) - \ln(s)$$

provided that $s > 0$ and $t \in \mathbb{R}$. *Hint:* $\displaystyle\int_0^t \sin(ux)du = \frac{1-\cos(tx)}{x}$.

6. Let $\mathcal{L}$ be the Lebesgue σ-field on $[0,\infty[$ and $d\mu(x) = e^{-x}dx$. Consider the linear subspace M of $H = L^2([0,\infty[,\mathcal{L},\mu)$ consisting of equivalence classes of functions that are periodic a.e. with period 2π, i.e.

$$f(x+2\pi) = f(x) \text{ a.a. } x \in [0,\infty[$$

(i) (4 points) Show that M is itself an L^2 space over a smaller σ-field than $\mathcal{L}$.
(ii) (4 points) Deduce that M is a *closed* linear subspace of H. What fact are you using here?
(iii) (4 points) Show that for $f, g \in H$,

$$\langle f, g\rangle = \sum_{k=0}^\infty e^{-2k\pi}\int_0^{2\pi} \overline{f(x+2k\pi)}g(x+2k\pi)e^{-x}dx$$

(iv) (4 points) Show that the closed linear span of the functions $x \mapsto e^{inx}$ as n runs over all integers is the whole of M. What fact are you using here?
(v) (4 points) For an arbitrary member f of H, let h be its orthogonal projection on M. Show that

$$h(x) = (1-e^{-2\pi})\sum_{k=0}^\infty e^{-2k\pi}f(x+2k\pi),$$

for almost all x in $[0,2\pi[$ (and extended by periodicity for other values of x).

7. Consider the trigonometric polynomials P_m and Q_m defined for nonnegative integers m inductively as follows

$$P_0(t) = Q_0(t) = 1 \text{ and } P_{m+1}(t) = P_m(t) + e^{i2^m t} Q_m(t),\ \ Q_{m+1}(t) = P_m(t) - e^{i2^m t} Q_m(t)$$

(i) (5 points) Show that $P_1(t) = 1 + e^{it}$, $Q_1(t) = 1 - e^{it}$, $P_2(t) = 1 + e^{it} + e^{2it} - e^{3it}$ and $Q_2(t) = 1 + e^{it} - e^{2it} + e^{3it}$.

(ii) (5 points) Show that $\widehat{P_m}(n) = 0$ if $n < 0$ or if $n > 2^m$ and that $\widehat{P_m}(n) = 1$ or -1 otherwise.

(iii) (5 points) Show that $|P_{m+1}(t)|^2 + |Q_{m+1}(t)|^2 = 2\left(|P_m(t)|^2 + |Q_m(t)|^2\right)$ and deduce first that $|P_m(t)|^2 + |Q_m(t)|^2 = 2^{m+1}$ for all t and then that $\sup_t |P_m(t)| \le 2^{\frac{m+1}{2}}$.

(iv) (5 points) Show that $\int_0^{2\pi} |P_m(t)|^2 dt = 2^{m+1}\pi$.

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