C04

McGILL UNIVERSITY

FACULTY OF SCIENCE

FINAL EXAMINATION

HONORS ANALYSIS 3, MATHEMATICS 354

Examiner: Professor Jakobson
Associate Examiner: Professor Toth

Date: Friday, December 14, 2012
Time: 14:00 - 17:00

INSTRUCTIONS

Answer all questions. Please give a detailed explanation for each answer. You may use any result proved in class or in the book, but must state precisely the statement that you are using.
Non-programmable calculators are permitted.
This is a closed-book exam
Dictionaries are permitted

Dmitry Jakobson

John A. Toth

This exam comprises the cover and two pages of questions.

Problem 1 (8 points).

a) (4 points) Let $f_n : [0,1] \to \mathbb{R}$ be a sequence of measurable functions on $[0,1]$. State measurability properties of $\liminf_{n\to\infty} f_n(x)$ and $\limsup_{n\to\infty} f_n(x)$. You don't need to prove these properties.

b) (4 points) Let $f_n : [0,1] \to \mathbb{R}$ be a sequence of measurable functions on $[0,1]$. Prove that the set $\{x : \lim_{n\to\infty} f_n(x) \textit{ exists}\}$ is Lebesgue measurable.

Problem 2 (8 points).

a) (4 points) State Monotone and Dominated convergence theorems. You don't need to prove them.

b) (4 points) Find the limit as $n \to \infty$

$$\int_0^n (1-(x/n))^n e^{x/2} dx$$

and justify your answer.

Problem 3 (8 points).

Given a measurable function $f :\in L^1([0,1])$, define a function $F_f : \mathbb{R}_+ \to \mathbb{R}_+$ (the *distribution function* of f) by

$$F_f(t) := \mu\{x : |f(x)| > t\}.$$

a) (4 points) Prove that

$$F_f(t) \leq \frac{||f||_1}{t}.$$

Hint: recall Chebyshev's inequality.

b) (4 points) Let $f = g + h$. Prove that $F_f(t) \leq F_g(t/2) + F_h(t/2)$.

Problem 4 (10 points).

a) (2 points) Define when a subset of a metric space is *connected.*

b) (2 points) Define when a subset of a metric space is *path connected.* What is the relationship between connected and path connected? You don't need to prove anything.

c) (2 points) State the *Intermediate Value theorem* for connected sets.

d) (4 points) Prove that the set B of $(x, y) \in \mathbb{R}^2$ such that $\{(x, y) : 1 \leq |x|+|y| \leq 2\}$ is connected. Prove that the function $g(x, y) = e^{x+y}$ attains the value 7 on the set B. You may use the fact that $2 < 2.7 < e < 2.8 < 3$.

Problem 5 (8 points).

a) (4 points) State Egorov's theorem.

b) (4 points) Verify the conclusion of Egorov's theorem for a sequence of functions $\{f_n\} : [0, 1] \to \mathbb{R}$ defined by $f_n(x) = \sin(1/(nx)), x > 0$ and $f_n(0) = 0$ for $n = 1, 2, \ldots$

Problem 6 (8 points).

a) (2 points) Define when a subset of a metric space is *closed.*

b) (2 points) State the properties of the closed sets under the union and intersection operations.

c) (4 points) Let X be a metric space and let $f : X \to \mathbb{R}$ and $g : X \to \mathbb{R}$ be continuous functions. Show that the set $\{x \in X : f(x) = g(x)\}$ is closed.