McGILL UNIVERSITY

FACULTY OF SCIENCE

FINAL EXAMINATION

MATHEMATICS 354

Examiner: Professor Jakobson | Date: Monday, December 10, 2010
Associate Examiner: Professor Jaksic | Time: 14:00 - 17:00

INSTRUCTIONS

Answer any all questions. Please give a detailed explanation for each answer. You may use any result proved in class or in the book, but must state precisely the statement that you are using.
Non-programmable calculators are permitted.
This is a closed-book exam
Dictionaries are permitted

This exam comprises the cover and two pages of questions.

Problem 1 (6 points).

Let $f(x), g(x) \in C([0,1])$. Assume that

$$\int_0^1 |f(x) - g(x)|^3 dx < 27,$$

and that

$$\int_0^1 |f(x)^2 + f(x)g(x) + g(x)^2|^{3/2} dx < 8.$$

Prove that

$$\int_0^1 |f(x)^3 - g(x)^3| dx \leq 12.$$

Problem 2 (8 points).

a) (2 points) Define when a subset of a metric space is *connected.*

b) (2 points) State the *Intermediate Value* theorem for metric spaces.

c) (2 points) Prove that the set B of $(x,y) \in \mathbb{R}^2$ such that $\{(x,y) : 1 \leq \sqrt{x^2+y^2} \leq 2\}$ is connected.

d) (2 points) Does the function $g(x,y) = \cosh(x^2+y^2)$ attain the value 15 on the set B? You may use the fact that $2 < 2.7 < e < 2.8 < 3$.

Problem 3 (8 points).

a) (2 points) Define when a subset of a metric space is *closed.*

c) (2 points) State basic properties of closed sets with respect to intersections and unions.

c) (4 points) Let $F_j(x_1, \dots, x_n), 1 \leq j \leq m$ be polynomials in n variables. Prove that the set of solutions of the system of algebraic equations

$$\begin{cases} F_1(x_1, \dots, x_n) = a_1, \\ \dots \\ F_m(x_1, \dots, x_n) = a_m. \end{cases}$$

is a closed subset of $\mathbb{R}^n$.

Problem 4 (10 points).

a) (2 points) Define when a subset of a metric space is *dense*.

b) (2 points) Define when a metric space is *separable*.

c) (3 points) Is l_2 separable? Prove or disprove.

d) (3 points) Is l_∞ separable? Prove or disprove.

Problem 5 (10 points). Let X be a normed linear space.

a) (2 points) State equivalent definitions of a continuous linear functional on X.

b) (2 points) Define the *norm* of a linear functional.

c) (3 points) Prove that the set of all bounded linear functionals on X themselves form a normed linear space (called the *conjugate space* X^* of X).

d) (3 points) Describe the conjugate space of l_2 (you don't need to prove that result). Give a formula for the norm of a linear functional on l_2, and prove it.

Problem 6 (8 points). Let X denote the space $C[0,1]$ with the d_∞ distance, $||f||_\infty = \max_{x\in[0,1]} |f(x)|$. Prove that the following operators T from X to itself are continuous, and find their norms.

a) $Tf(x) = f(x^\alpha), \alpha > 0$.

b) $Tf(x) = \int_0^x f(t)dt$.