McGILL UNIVERSITY

FACULTY OF SCIENCE

FINAL EXAMINATION

MATHEMATICS 354

Examiner: Professor Jakobson
Associate Examiner: Professor Klemes

Date: Monday, December 18, 2006
Time: 14:00 - 17:00

INSTRUCTIONS

Answer any 6 of the following 7 questions. Each question is worth 10 points. Please give a detailed explanation for each answer. You may use any result proved in class or in the book, but must state precisely the statement that you are using.
Non-programmable calculators are permitted.
This is a closed-book exam
Dictionaries are permitted

This exam comprises the cover and two pages of questions.

This exam is printed on both sides (double-sided)

Problem 1. (10 points) Let X be a metric space with the distance ρ, and Y be a metric space with the distance σ. Let $X \times Y$ be the set of pairs $\{(x, y) : x \in X, y \in Y\}$. Define a distance d on $X \times Y$ by

$$d((x_1, y_1), (x_2, y_2)) = \max\{\rho(x_1, x_2), \sigma(y_1, y_2)\}.$$

Note that this defines, e.g., the d_∞ distance on $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$.

a) (3 points) Prove that d defines a distance on $X \times Y$.

b) (3 points) Define when a metric space is *totally bounded.*

c) (4 points) Prove that if (X, ρ) and (Y, σ) are totally bounded, then so is $(X \times Y, d)$.

Problem 2. (10 points)

a) (3 points) Define when a family of functions is *(uniformly) equicontinuous.*

b) (3 points) State the *Arzela-Ascoli* theorem.

c) (4 points) Consider the space $X = C([0, 1])$ with the supremum distance, $d(f, g) = \sup_{x\in[0,1]} |f(x) - g(x)|$, induced by the norm $||f|| = \sup_{x\in[0,1]} |f(x)|$. Let Y be a bounded subset of X. Prove that the set of functions

$$F(x) = \int_0^x t \cdot f(t)dt, \qquad f \in Y$$

has compact closure.

Problem 3. (10 points)

a) (2 points) Define when a subset of a metric space is *compact.*

b) (3 points) Give an equivalent definition of compactness in $\mathbb{R}^n$.

c) (5 points) Suppose that the function $f : \mathbb{R}^n \to \mathbb{R}$ is continuous, and $f(\mathbf{x}) \geq ||\mathbf{x}||$ for all $\mathbf{x} \in \mathbb{R}^n$. Prove that $f^{-1}([0, 1])$ is compact.

Problem 4. (10 points)

a) (2 points) Define when a subset of a metric space is *connected.*

b) (2 points) State the *Intermediate Value* theorem for metric spaces.

c) (3 points) Prove that the set B of $(x, y) \in \mathbb{R}^2$ such that $(x-1)^2 + (y-2)^2 \leq 5$ is connected.

b) (3 points) Does the function $g(x, y) = \exp(|x| + |y|)$ attain the value 50 on the set B? You may use the fact that $2 < 2.7 < e < 2.8 < 3$.

Problem 5. (10 points)

a) (2 points) State the *Contraction Mapping* theorem.

b) (6 points) Let Y be the set of continuous functions on $[0, 1]$ that take values in $[0, 1]$, with the uniform distance. Prove that Y is complete. Also, prove that the mapping A defined on Y by the formula $[Af](x) = [(f(x))^3 + x + 2]/4$ is a contraction mapping of Y into itself.

c) (2 points) Conclude that there exists a unique function $f : [0, 1] \to [0, 1]$ satisfying $f(x) = [(f(x))^3 + x + 2]/4$.

Problem 6. (10 points)

a) (4 points) Define when a sequence of functions (from a metric space to another metric space) *converges uniformly.*

b) (6 points) Consider the sequence of functions $f_n : [0, \pi] \to \mathbb{R}$ defined by the formula $f_n = \sin \circ \sin \circ \ldots \circ \sin$ (taken n times), i.e. $f_1(x) = \sin x$, $f_2(x) = \sin(\sin x)$, $f_3(x) = \sin(\sin(\sin x))$ etc. Prove that the sequence of functions $\{f_n\}$ converges uniformly to the zero function as $n \to \infty$. You can use the fact that $0 < \sin x < x$ for $0 < x < \pi$.

Problem 7. (10 points)

a) (4 points) State the *Implicit Function* theorem.

b) (6 points) Let x, y, u, v be related by

$$xe^{u+v} + 2uv - 1 = 0, \qquad ye^{u-v} - \frac{u}{1+v} - 2x = 0.$$

Compute partial derivatives $(\partial u/\partial x), (\partial v/\partial y)$ at the point where $x = 1$, $y = 2$ and $u = v = 0$.