Department of Mathematics and Statistics
McGill University

FINAL EXAMINATION
Math354, Honours Analysis 3

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Date & time: Monday December 12, 2005, 14:00-17:00

Instructions:

- This is a closed-book exam.
- You are allowed to use non–programmable calculators.
- Give detailed and complete solutions.
- Attempt all problems.
- Write your answers in the exam booklet.
- The use of a regular dictionary is allowed.

This exam consists of a cover sheet plus one page containing eight questions.

1. **[5pt]** Let X be a separable metric space and let $Y \subseteq X$. Show that Y is separable as well.

2. **[5pt]** Let $k : [0,1] \times [0,1] \to \mathbb{R}$ be a continuous function and consider the map $K : C([0,1],\mathbb{R}) \to C([0,1],\mathbb{R})$ given by

$$(Kf)(x) = \int_0^1 k(x,y)f(y)\mathrm{d}y.$$

Show that any sequence $f_n \in C([0,1],\mathbb{R})$ satisfying $\|f_n\| \leq 1$ has a subsequence f_{n_j} with Kf_{n_j} uniformly convergent.

(Suggestion: Use the Ascoli–Arzela theorem.)

3. **[6pt]** Let X and Y be compact metric spaces. Show that $X \times Y$ is compact in two ways:

a) by using sequential compactness,

b) by using completeness and total boundedness.

4. **[4pt]** Let $\{A_n\}$ be a collection of connected subsets of a metric space X, such that $A_n \cap A_{n+1} \neq \emptyset$ for all n. Show that $\cup_n A_n$ is connected.

5. **[4pt]** Let $X = \{(0,0)\} \cup \{(x, \sin(x)\sin(1/x)) \mid 0 < x \leq 1\} \subset \mathbb{R}^2$. Is X path–connected? Justify your answer!

6. **[5pt]** Let $L \in BL(\mathbb{R}^n)$ be an invertible map, and let $g \in C^1(\mathbb{R}^n)$ be such that $\|g(x)\| \leq M\|x\|^2$. Show that $f(x) = Lx + g(x)$ is locally invertible near 0.

7. **[5pt]** Let f be a map of class C^1 on a Banach space X such that $f(tx) = tf(x)$ for all real t and all $x \in X$. Show that f is linear, and in fact that $f(x) = Df(0)x$.

8. **[6pt]** Show that the system

$$\begin{aligned} xy^2 + xzu + yv^2 &= 3 \\ u^3yz + 2xv - u^2v^2 &= 2 \end{aligned}$$

has a C^∞ solution $u(x,y,z)$, $v(x,y,z)$ near $(x,y,z) = (1,1,1)$, $(u,v) = (1,1)$. Find $\frac{\partial}{\partial y}v(1,1,1)$.