McGill UNIVERSITY

FACULTY OF SCIENCE

FINAL EXAM

MATH 354

ANALYSIS III

Examiner: Professor K.N GowriSankaran
Associate Examiner: Professor S.Drury

Date: Thrusday December 9, 2004
Time: 2:00 p.m -5:00 p.m

INSTRUCTIONS

1. Please attempt to answer all 6 questions for full credit..
2. Write your answers in the exam booklets provided.
3. This is a closed book exam.
4. No calculators are permitted
5. This exam consists of the cover page and 2 pages of 6 questions .

1. (a) Define the notion: f is uniformly continuous from a metric space with values in another metric space.

   (b) If $f : (X, d) \to (Y, \rho)$ is uniformly continuous and $x_n \epsilon X$ forms a Cauchy sequence, show that $(f(x_n))$ is Cauchy.

   (c) Suppose $A \subset X$ is dense and $f : A \to \mathbb{R}$ is uniformly continuous, show that there exists a unique uniformly continuous function F on X such that $F|_A = f$

   (d) Suppose X is a compact space and $A \subset X$ a dense subset and $f : A \to \mathbb{R}$. Prove that f is uniformly continuous on A if and only if f has a continuous extension to X.

2. Suppose $\{A_i\}_{i\epsilon I}$ is a locally finite family in a metric space, i.e for every point $x \epsilon X$, there is an open ball $B(x, r_x)$ of radius $r_x > 0$ such that $B(x, r_x) \cap A_i = \emptyset$ except for $i\epsilon I_x$, I_x a finite subset of I. Suppose $K \subset X$ is compact. Prove that $\{i\epsilon I : A_i \cap K \neq \emptyset\}$is a finite set.

3. (a) Let X be a connected metric space and $f : X \to Y$ a continous mapping onto another metric space Y. Prove that Y is connected.

   (b) Suppose A and B are connected subsets of a metric space X such that $(\overline{A} \cap B) \cup (A \cap \overline{B}) \neq \emptyset$. Show that $A \cup B$ is connected.

4. Suppose $\{f_n\}$ is a sequence of real valued continuous functions on $\mathbb{R}^3$ satisfying (1) $\{f_n\}$ is equi-continuous at each point and (2) $\{f_n(\vec{x})\}$ is a convergent sequence of real numbers for each $\vec{x}$. Prove that $\{f_n\}$ converges uniformly on every bounded subset of $\mathbb{R}^3$.

5. (a)State the Stone-Weierstrass Theorem.

   (b) A continuous function $f : [0,1] \to \mathbb{R}$ is such that $\int_0^1 x^n f(x)dx = 0$ for all $n = 0, 1, 2, 3 \ldots$ Prove that $f \equiv 0$

6. (a) Define the derivative $\vec{f}'(\vec{x})$ of a function $\vec{f} : V$ an open set $\subset \mathbb{R}^n \to \mathbb{R}^m$

   (b)$\vec{f}$ is a differentiable function defined on an open connected set of $V \subset \mathbb{R}^n (V \neq \emptyset)$ with the values in $\mathbb{R}^m$. If $\vec{f}' \equiv \vec{0}$, prove that $\vec{f} \equiv constant.$

   (c)Suppose $\vec{f}$ is defined on an open set $W \subset \mathbb{R}^n$ with values in $\mathbb{R}^m$ and is such that $\vec{f}'$ (exists and) $\equiv \vec{0}$ on W. Prove that $\vec{f}$ takes at the most countably many different values.

   (d) Give an example of a function as in (c) above taking infinitely many different values.