McGILL UNIVERSITY

FACULTY OF SCIENCE

FINAL EXAMINATION

MATH 338
HISTORY AND PHILOSOPHY OF MATH

Examiner: Professor M.Makkai
Associate Examiner: Professor J. Loveys

Date: Thursday December 20, 2007
Time: 9:00 AM- 12:00 PM

INSTRUCTIONS

1. Please answer questions in the exam booklets provided.
2. Please read the instructions on page 1 of this exam carefully before you begin to write this exam.
3. This is a closed book exam.
4. Faculty approved calculators permitted only.
5. Use of a regular and or translation dictionary is permitted.

This exam comprises of the cover page,one page of instructions (see page 1) and 4 pages of questions.

Instructions

1. **1)** Write the essay [E];
2) choose **two (2)** of the questions from **Group 1: [1] to [4]** to answer;
3) choose **two (2)** of the questions from **Group 2: [5] to [8]** to answer.

If more questions are attempted, the best two answers of each of Group 1 and Group 2 will be counted towards the exam grade.

Restriction: *The essay* [E] *and the answers to questions from groups 2) and 3) may not overlap explicitly.*

You may write an essay on a subject and do a question on a related subject, but if you do that, material that is given in essentially the same way in both treatments will be counted towards your exam grade only once. You may mention the same concepts in the essay and in the question, but any details that the question requires you to provide should be avoided in the essay.

You cannot earn marks twice for the same work. On the other hand, if a context in the essay requires it, you may make a reference to your answer to a question in the style "for details, see question [?]".

2. *Faculty approved calculators* are allowed.

3. This is a closed-book exam. It is not allowed to use a computer to obtain information or to get direct answers to complex questions.

4. Justify all your answers. Results may be checked by a calculator directly, but they have to be obtained by calculations that follow the mathematics involved. *Results that are not properly justified may earn reduced or no marks even if they are correct.*

5. If you do any bonus questions, make sure you do not spend too much time on them. *Bonus marks will be limited.*

[E](30%) Write an essay of approximately 800 to 1,000 words on one of the following subjects:

- **(1)** Trigonometry
- **(2)** Conic sections and coordinate geometry
- **(3)** Complex numbers and cubic equations.

The essay should be informative both historically and mathematically.

The material treated in the essay should come, in essence, from the material covered in the text or in the classroom lectures and the additional notes provided for the course. Material that is historically or mathematically foreign in spirit to what was done in the course will not be graded.

Group 1: questions [1], [2], [3] and [4]. Answer two (2) questions in Group 1.

[1](17.5%) **(i)** (*Egyptian fractions*) **Write** $\frac{49}{300}$ as a sum of unit fractions (a unit fraction is a number of the form $\frac{1}{p}$ with p an integer greater than 1) in two different ways: **first**, by the so-called greedy algorithm; and **secondly**, by writing the numerator 49 as a sum of distinct powers of 2, and using a suitable formula for fractions of the form $\frac{2}{q}$.

(ii) (*Anthyphairesis*) Let n be any integer at least 1, and let $x=\sqrt{(n^2+2)}$. **Give** the (infinite periodic) continued fraction expansion

$$[n_0, n_1, n_2, n_3, \ldots] = n_0 + \cfrac{1}{n_1 + \cfrac{1}{n_2 + \cfrac{1}{n_3 + \cfrac{1}{\ldots}}}}$$

of x: determine each n_k in terms of the given n.

Hint: prove and use the fact that $n < \sqrt{(n^2+2)} < n+1$.

(iii) Using appropriate formulas, or directly, **compute** the convergents $[n_0]$, $[n_0, n_1]$, $[n_0, n_1, n_2]$ and (only) the denominator of the convergent $[n_0, n_1, n_2, n_3]$ of the expansion in (ii), and **conclude** that

$$0 < \sqrt{(n^2+2)} - (n+\frac{2n}{2n^2+1}) < \frac{1}{n\cdot(2n^2+1)\cdot(2n^2+2)}.$$

[2](17.5%) *(Babylonian follies)*

(i) **Give** a method of producing infinitely many quadruples (a, b, d, e) of positive integers a, b, d, e such that

the right triangle with sides a and b, and the right triangle with sides d and e have the same (length of the) hypotenuse;

and moreover, "to avoid cheating",

the numbers a, b, d, e are (pairwise) distinct,

and

there is no prime number dividing all of them.

(ii) **Apply** the method to obtain two quadruples (a_1, b_1, d_1, e_1), (a_2, b_2, d_2, e_2) satisfying the conditions in (i), and also such that every one of the eight numbers involved is at least 100, and each one of the eight is different from any other.

[3](17.5%) *(The Babylonian/Newtonian method of calculating square-roots, and other roots)*

(i) Describe the iterative method of approximating $\sqrt{N}$ for a non-square integer N that the Babylonians might have used ("One possible method for which there is some textual evidence"). Include an error estimate in your description. Prove your assertions.

(ii) To approximate $\sqrt{7}$, start with the value $a_0=3>\sqrt{7}$, and calculate the fourth approximation a_4 to $\sqrt{7}$. To see some nice things happening with the numbers, calculate each quantity involved in the form of a simple fraction $\frac{r}{s}$ $(r, s\in\mathbb{N})$, and keep the denominators factored. How good an approximation of $\sqrt{7}$ is a_4 ?

(iii)(for bonus points) (a) Can you see a regularity here that will persist for all stages of the iteration?
(b) One can also calculate $\sqrt{7}$ by anthyphairesis. Can you comment about how the speed of convergence of the "Babylonian" method compares to that of anthyphairesis?

[4](17.5%) State Eudoxos' definition of equality of ratios, and using it, prove the law of similarity: in two triangles with pairwise equal angles, the sides of one triangle are to each other as the corresponding sides of the other triangle.

Group 2: questions [5], [6], [7] and [8]. Answer two (2) questions in Group 2.

[5](17.5%) There are four points in space: A, B, X and Y. Points A and B are on a perfectly flat horizontal plane; X and Y are in space above the plane. Points X and Y are not physically accessible, but they are visible from A and B. Each of the points A and B is clearly visible from the other. The length $\overline{AB}$ (distance between A and B) can be measured directly; but no other distance between any two of the four points can be measured directly. Angles enclosed by lines of sight from an accessible point can be measured.

(i) Introduce short notation for angles and distances involved, and, using trigonometry, **give a formula** for the length $\overline{XY}$ (distance between X and Y) in terms of $\overline{AB}$ and angles that can be measured directly.

Derive your formula carefully by referring to laws of trigonometry. Do not attempt proving any of the laws used.

(ii) Choose the value 100 for $\overline{AB}$, choose different acute angles between 40° and 60° as values for the angles you are using in the formula in (i), and, using the calculator, compute the value of $\overline{XY}$.

(iii)(For bonus points) Discuss the trigonometry of four points A, B, X and Y in general position in space. State and/or prove some of the laws that hold true for angles and distances involved, laws that go beyond the usual laws that hold in a triangle in a plane.

[6](17.5%) **(i)** **Give** a careful and complete **derivation** of the standard equation ($\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$) of the ellipse in a suitably chosen Cartesian coordinate system, from (the restricted) Apollonius' definition of the ellipse (which says that an ellipse is the intersection of a *right* cone with a suitably positioned plane). Use figures to help explain the proof.

Derivation of the equation from a *different* definition of the ellipse (for instance, the usual one of the form " $\overline{PF}_1+\overline{PF}_2=2a$ ") will *not* earn *any* marks.

(ii) Define the foci of the ellipse, and prove the formula $\overline{PF}_1+\overline{PF}_2=2a$ (where F_1, F_2 are the foci of the ellipse, P is an arbitrary point on the ellipse).

[7](17.5%) **(i) State and prove** Ptolemy's theorem concerning a quadrilateral (quadrangle) inscribed in a circle.

The proof should be based on elementary geometry (geometry of Euclid's "Elements"). Other proofs such as ones that use trigonometry or coordinate geometry will *not* earn *any* marks.

(ii) Derive the addition theorems for the trigonometric functions $\sin(\alpha)$ and $\cos(\alpha)$ from Ptolemy's theorem.

[8](17.5%) **(i)** **Give** Hamilton's formal construction of the complex numbers and the operations of addition and multiplication on them.

(ii) **Give a proof** of the associative law for multiplication of complex numbers *directly* from Hamilton's definition.

(iii) **Identify** the imaginary unit $\mathbf{i}$ $(=\sqrt{-1})$ and the real numbers with their counterparts in Hamilton's definition, and **show** that every complex number can be written as $x+\mathbf{i}\cdot y$ with suitable reals x and y.

(iv) **Prove** that for any complex number $z\neq 0$, there is another complex number w such that $z\cdot w=1$.

(v) **State and prove** De Moivre's formula (involving the "modulus" and the "argument" of complex numbers) expressing the product $z\cdot w$ of complex numbers z and w.