

Student Name:
Student Id#:

McGILL UNIVERSITY

FACULTY OF SCIENCE

FINAL EXAMINATION

MATH 326/376

Non Linear Dynamics and Chaos and Honors Nonlinear Dynamics and Chaos

Examiner: Professor A. Humphries
Associate Examiner: Professor G. Tsogtgerel

Date: Wednesday December 15, 2010
Time: 9:00 a.m - 12.00 p.m

INSTRUCTIONS

1. **Students in MATH 326 answer 5 of the first 7 questions. If you answer more than 5 questions, credit will be given for the best 5 answers. Do not answer question 8.**
2. **Students in MATH 376 answer 5 of the last 6 questions. If you answer all of these questions, credit will be given for the best 5 answers. Do not answer questions 1 or 2.**
3. Please answer all questions in the exam booklets provided, starting each question on a new page.
4. All questions carry equal weight.
5. This is a closed book exam. Notes and textbooks are not permitted.
6. Translation dictionaries (English-French) are permitted.
7. Calculators, including graphical calculators are permitted.
8. This exam comprises of the cover page and 3 pages of 8 questions.

1. (Do not do this question if you are a math376 student)
 - (a) Sketch phase portraits (but do *not* attempt to write down equations defining the dynamical systems) showing that it is possible for a dynamical system to have a fixed point which
 - i. is Lyapunov stable but not attracting,
 - ii. is attracting but not Lyapunov stable,
 - iii. is a saddle point with a homoclinic connection. In this case label the stable and unstable manifolds of the fixed point.
 - (b) Give examples of functions $f(u, \mu)$ such that the system $\dot{u} = f(u, \mu)$ has bifurcations of the following type at $u = 0, \mu = 0$:
 - i. saddle node,
 - ii. transcritical,
 - iii. supercritical pitchfork.

In each case sketch the corresponding bifurcation diagram, indicating the stability of the fixed points (you do not need to justify your answer).

2. (Do not do this question if you are a math376 student)
Consider a dynamical system defined by

$$\dot{x} = x(3 - x - 2y),$$

$$\dot{y} = y(3 - 2x - y),$$

where $x(t), y(t)$ represent the size of two different competing animal populations at time t ; and so we only consider $x, y \geq 0$.

- (a) Find all the fixed points of this dynamical system which satisfy $x, y \geq 0$, and determine their linear stability types.
 - (b) Sketch the phase portrait for $x, y \geq 0$, clearly indicating the fixed points, isoclines (or nullclines) and the direction of flow. Label the *stable* and *unstable* manifolds of any saddle points. (You do *not* need to find or use the eigenvectors of the Jacobian at the fixed points).
3. (a) Find a bifurcation from the fixed point $u = 0$ in the dynamical system

$$\dot{u} = \mu u - \frac{u^3}{1 + u^2}, \quad u \in \mathbb{R},$$

as the parameter $\mu \in \mathbb{R}$ is varied. What type of bifurcation is observed? Sketch the bifurcation diagram, indicating stability of the fixed points (except at the bifurcation point).

- (b) Consider the dynamical system

$$\dot{u} = \mu - u - e^{-u}, \quad u \in \mathbb{R},$$

with parameter $\mu \in \mathbb{R}$. Find a formula to determine μ so that the point u is a fixed point. Hence find the bifurcation point, and sketch the bifurcation diagram, indicating stability of the fixed points (except at the bifurcation point).

4. Consider the dynamical system on a circle

$$\dot{\theta} = \mu \sin \theta - \sin^3 \theta, \quad \theta \in [0, 2\pi).$$

- Find two fixed points $\theta \in [0, 2\pi)$ which exist for all values of μ .
- Determine the stability of the fixed points just found for all values of $\mu \in \mathbb{R}$, and thus identify two possible bifurcation points.
- Find all the other fixed points for the dynamical system, as μ is varied.
- Sketch the bifurcation diagram for this dynamical system indicating the stability of all the fixed points, and the types and approximate locations of all the bifurcations.
- Sketch a phase portrait for the dynamical system when the parameter μ is in the region where the dynamical system has most fixed points.

5. (a) Consider the dynamical system

$$\begin{aligned} \dot{x} &= -x^3 + xy^2, \\ \dot{y} &= -y^3. \end{aligned}$$

- What does linearization tell you about the stability of the fixed point at the origin?
- Using the Lyapunov functional, $V(x, y) = x^2 + y^2$, or otherwise, show that the fixed point is asymptotically stable.
- Sketch the phase portrait.

(b) Consider the dynamical system

$$\begin{aligned} \dot{x} &= -y + xy^3, \\ \dot{y} &= x - y^2, \end{aligned}$$

which has fixed points at $(0, 0)$, $(1, -1)$ and $(1, 1)$.

- What does it mean for a dynamical system to be reversible? Show that this system is reversible.
- What is the stability type of the fixed point at the origin?
- Sketch a plausible phase portrait.

6. Consider the system of differential equations

$$\begin{aligned} \dot{x} &= x - y + 2xy - x(x^2 + y^2), \\ \dot{y} &= x + y + 2y^2 - y(x^2 + y^2). \end{aligned}$$

- Determine the type and stability of the fixed point at $(0, 0)$.
- Convert the differential equations to polar coordinates.
- Show that there is an invariant annulus in the (x, y) plane, such that solutions may enter the annulus but not leave.
- Use the Poincaré Bendixson theorem to deduce the existence of a periodic orbit for this dynamical system.
- Sketch a plausible phase portrait of this dynamical system.

7. Consider the system of differential equations

$$\begin{aligned}\dot{x} &= \mu x - xy, & x(0) &= x_0, \\ \dot{y} &= x^2 - y, & y(0) &= y_0,\end{aligned}$$

where $\mu \in \mathbb{R}$ is a bifurcation parameter.

- (a) Investigate the stability of the fixed point at $(0, 0)$ and hence identify a candidate bifurcation point.
- (b) Confirm that a bifurcation occurs by solving for all the fixed points of the dynamical system for all $\mu \in \mathbb{R}$. What type of bifurcation occurs?
- (c) Show that the eigenvalues, λ , of the Jacobian matrix at the non-zero fixed point(s) satisfy

$$\lambda^2 + \lambda + 2\mu = 0,$$

and hence show that such fixed points are always stable.

- (d) Use Dulac's criterion with weight function $g(x, y) = 1/x$ to deduce that any periodic orbit must cross the line $x = 0$, and hence that the dynamical system has no periodic orbits.
8. (Do not do this question if you are a math326 student)

- (a) Consider the differential equation

$$\dot{u} = u^{1/3}, \quad u(0) = 0.$$

- i. Find a solution to this problem for $t \geq 0$.
 - ii. Find another solution to this problem for $t \geq 0$.
 - iii. Show that there are infinitely many solutions to this problem.
- (b) Consider the differential equation

$$\dot{x} = f(x, r) = rx(1 - x) - x,$$

where r is a bifurcation parameter.

- i. Solve for all the fixed points, and hence identify a bifurcation which occurs at $x = 0$.
- ii. Show that by defining a new bifurcation parameter μ , in a suitable way, that the system can be translated so that the bifurcation occurs at $(x, \mu) = (0, 0)$.
- iii. Show that the system can be written as

$$\dot{y} = \mu y - y^2 + \mathcal{O}(x^3) = \mu y - y^2 + \mathcal{O}(y^3),$$

by making the near identity change of variables $y = x + cx^2$ for suitable choice of c (which you should determine).