Student Name:
Student Id#:

McGILL UNIVERSITY

FACULTY OF SCIENCE

FINAL EXAMINATION

MATH 326/376

Non Linear Dynamics and Chaos and Honors Nonlinear Dynamics and Chaos

Examiner: Professor A. Humphries
Associate Examiner: Professor D. Jakobson

Date: Thursday December 18, 2008
Time: 2:00 p.m - 5.00 p.m

INSTRUCTIONS

1. **Students in MATH 326 answer any 6 questions.**
2. **Students in MATH 376 answer questions 3 through 8.**
3. Please answer all questions in the exam booklets provided, starting each question on a new page.
4. All questions carry equal weight.
5. This is a closed book exam. Notes and textbooks are not permitted.
6. Translation dictionaries (English-French) are permitted.
7. Calculators, including graphical calculators are permitted.
8. This exam comprises of the cover page and 3 pages of 8 questions.

1. (Do not do this question if you are a math376 student)

   (a) Sketch phase portraits (but do *not* attempt to write down equations defining the dynamical systems) showing that it is possible for a dynamical system to have a fixed point which

      i. is Lyapunov stable but not attracting,

      ii. is attracting but not Lyapunov stable,

      iii. is a saddle point with a homoclinic connection. In this case label the stable and unstable manifolds of the fixed point, and the homoclinic orbit.

   (b) Consider the dynamical system

   $$\dot{u} = Au, \quad u \in \mathbb{R}^2,$$

   where A is a 2×2 matrix. Give examples of A where the fixed point at the origin is a

      i. saddle

      ii. (linear) centre

      iii. unstable node

      iv. stable focus (stable spiral).

      v. stable star

   You do not need to justify your answer.

2. (Do not do this question if you are a math376 student) Consider the system

   $$\dot{u} = \mu u - u^2 + u^3, \qquad x \in \mathbb{R}$$

   where μ is a real parameter.

   (a) Find all the bifurcation points for this system, and state the type of each bifurcation.

   (b) Sketch a bifurcation diagram, indicating the stability of the fixed points, and the locations of the bifurcations.

   (c) Sketch phase portraits for three different values of μ, say $\mu_1 < \mu_2 < \mu_3$, such that there is not a bifurcation at μ_i but there is a bifurcation between each value of μ_i.

3. Let

   $$H(x, y) = \frac{1}{2}x^2 + \frac{1}{2}y^2 - \frac{1}{4}x^4.$$

   (a) Consider the two-dimensional Hamiltonian system

   $$\dot{x} = \frac{\partial H}{\partial y}, \qquad \dot{y} = -\frac{\partial H}{\partial x}$$

   where H is defined above. Find all the fixed points of this dynamical system and determine their stability types. Sketch a phase portrait, and label the stable and unstable manifolds of any saddle points.

   (b) Consider the general two-dimensional gradient system

   $$\dot{x} = -\frac{\partial H}{\partial x}, \qquad \dot{y} = -\frac{\partial H}{\partial y}$$

   where H is defined above. Find all the fixed points of this dynamical system and determine their stability types. Sketch a phase portrait, and label the stable and unstable manifolds of any saddle points.

4. (a) Consider the dynamical system

$$\dot{x} = -x^3 + xy + xy^2,$$

$$\dot{y} = -y^3 - x^2 + x^2 y.$$

   i. What does linearization tell you about the stability of the fixed point at the origin?
   ii. Using the Lyapunov functional, $V(x, y) = x^2 + y^2$, or otherwise, show that the fixed point is asymptotically stable.
   iii. Sketch the phase portrait.

   (b) Consider the differential equation

$$\dot{u} = u^{1/3}, \quad u(0) = 0.$$

   i. Find a solution to this problem for $t \geqslant 0$.
   ii. Find another solution to this problem for $t \geqslant 0$.
   iii. Show that there are infinitely many solutions to this problem.

5. Consider the dynamical system

$$\dot{x} = \mu x - \sin x, \qquad x \in \mathbb{R}$$

where $\mu > 0$ is a parameter.

   (a) Show that this dynamical system has exactly one fixed point for all μ sufficiently large.
   (b) Show (graphically or otherwise) that there are infinitely many bifurcations for $\mu > 0$.

Let $0 < \ldots \mu_4 < \mu_3 < \mu_2 < \mu_1$ be the values of μ for which bifurcations occur. So μ_1 be the largest value of $\mu > 0$ at which a bifurcation occurs, μ_2 is the next largest value, and no bifurcations occur for $\mu \in (\mu_2, \mu_1)$, etc.

   (c) What is μ_1? What types of bifurcation occur at $\mu = \mu_1$ and at $\mu = \mu_2$?
   (d) Sketch two phase portraits. One for $\mu \in (\mu_2, \mu_1)$ and one for $\mu \in (\mu_3, \mu_2)$.
   (e) Sketch the bifurcation diagram for $\mu > 0$ and $x \in [-4\pi, 4\pi]$, indicating the stability of the fixed points. (You do *not* need to find the exact location of the bifurcation at μ_2.)

6. Consider the dynamical system in polar coordinates

$$\dot{r} = r(6 + r\mu \sin\theta - r^2),$$

$$\dot{\theta} = r^2 - 5r + 4.$$

where $\mu \in \mathbb{R}$ is a parameter.

   (a) Find a periodic orbit when $\mu = 0$. What is the period of this orbit?
   (b) Now consider the system with $\mu = 1$. Show that $\dot{r} > 0$ for $r < 2$ and $\dot{r} < 0$ for $r > 3$ and hence deduce that the system has a periodic orbit, and sketch a plausible phase portrait. State (but do not prove) any theorem(s) you need to justify the existence of a periodic orbit in this case.

7. Consider the dynamical system $\dot{u} = f(u)$ where $u = (x, y)$ and

$$\dot{x} = x(\mu - x - y), \quad x \geqslant 0,$$

$$\dot{y} = y(x - 1), \quad y \geqslant 0,$$

where $x \geqslant 0$ represents a population of a prey animal and $y \geqslant 0$ represents a predator, and $\mu > 0$ is a positive parameter.

   (a) Find all the fixed points for $x \geqslant 0$, $y \geqslant 0$ and their dependence on $\mu > 0$, and determine their linear stability types.

   (b) At what value of $\mu > 0$ does a bifurcation occur? State the bifurcation type.

   (c) Sketch two plausible phase portraits for $\mu > 0$, one for μ each side of the bifurcation point.

8. Consider the system of differential equations

$$\dot{x} = \mu x - y - x(x^2 + y^2),$$

$$\dot{y} = x + \mu y - 2y(x^2 + y^2).$$

where $\mu \in \mathbb{R}$ is a parameter.

   (a) Using the Lyapunov functional $V(x, y) = x^2 + y^2$, or otherwise, s how that all solutions satisfy $\lim_{t\to\infty}(x(t), y(t)) = (0, 0)$ when $\mu < 0$ and when $\mu = 0$.

   (b) By finding the eigenvalues of the Jacobian matrix at the fixed point $(0, 0)$, find a bifurcation that occurs as μ is varied. What type of bifurcation is observed? Is it supercritical, subcritical or degenerate?

   (c) Sketch two plausible phase portraits, for the region of phase space near to $(0, 0)$, one for μ less than but close to the bifurcation value, and one for μ greater than but close to the bifurcation value.