McGILL UNIVERSITY

FACULTY OF SCIENCE

FINAL EXAMINATION

MATH 325

HONOURS ORDINARY DIFFERENTIAL EQUATIONS

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Date: Tuesday December 7, 2010
Time: 9:00AM - 12:00 PM

INSTRUCTIONS

1. Answer all questions in the exam booklets provided. Start each question on a new page.
2. All questions carry equal weight.
3. This is a closed book exam. No crib sheets, textbooks or any other aids are permitted.
4. Calculators are permitted.
5. Dictionaries are not permitted.

This exam comprises the cover page, 2 pages of 6 questions and a table of Laplace Transforms.

1. (a) Solve (implicitly)

$$(3x^2y + 2xy + y^3)dx + (x^2 + y^2)dy = 0,$$

by finding an integrating factor $\mu(x)$ (which is a function of x only).

(b) Find the general solution $y(x)$ of

$$y^{(4)} - 3y'' - 4y = x + e^{2x}.$$

2. (a) Let

$$y'(t) = y(t)^2 - 3, \qquad y(0) = 2,$$

and let $y_0(t) = 2$ for all $t > 0$. Find the first two approximations $y_1(t)$ and $y_2(t)$ to the exact solution by Picard iteration.

(b) Show that

$$y'(t) = y(t)^{1/3}, \quad y(0) = 0,$$

has more than one solution, by finding two different solutions. (Briefly), which of the conditions of the basic existence and uniqueness theorem for solutions of the initial value problem is violated in this example?

3. (a) Let $p(x)$, $q(x)$ and $g(x)$ be continuous on an interval I, let $y_1(x)$, $y_2(x)$ be a set of fundamental solutions to $L[y] = 0$ on I, where $L[y] := y'' + p(x)y' + q(x)y$, and let $y_p(x)$ be a particular solution to $L[y](x) = g(x)$ on I. Show that there is a unique choice of c_1, c_2 such that $y(x) = c_1y_1(x) + c_2y_2(x) + y_p(x)$ solves the initial value problem $L[y](x) = g(x)$ and $y(x_0) = \alpha$, $y'(x_0) = \beta$, (where $x_0 \in I$).

(b) Solve

$$y'' - 2y' + y = x^2, \qquad y(0) = 1, \quad y'(0) = 0.$$

4. Let

$$L[y] = (x - 2)y'' + (1 - x)y' + y.$$

(a) Show that $y_1(x) = e^x$ solves the homogeneous problem $L[y_1](x) = 0$.

(b) Find second solution $y_2(x)$ which solves $L[y_2](x) = 0$ (with y_2 linearly independent of y_1).

(c) Let y_1, y_2 be a fundamental set of solutions to $L[y](x) = 0$. State (but do not derive) the equations which define a particular solution $y_p(x)$ which solves $L[y_p](x) = g(x)$ when using Variation of Parameters.

(d) Find the general solution of

$$L[y](x) = (x - 2)^2e^{2x},$$

where $L[y]$ is the differential operator defined above.

5. Consider the differential equation

$$4xy'' + 2y' + y = 0$$

   (a) Define *regular singular point.* Find the regular singular point of the given equation, state the indical equation and find its roots.

   (b) Find a fundamental set of Frobenius series solutions for $x > x_0$ expanded about the regular singular point x_0. Your answer should include an expression for the general nth coefficient in each series solution, not just a recursion relation.

   (c) For what value(s) of α is it possible to satisfy the initial value problem $y(x_0) = 1$, $y'(x_0) = \alpha$ for this equation?

6. Let $y(t)$ solve

$$y'' + y = \delta(t - \pi) + \mathcal{U}(t - 2\pi)e^{-(t-2\pi)}, \qquad y(0) = 1, \quad y'(0) = 0.$$

   (a) Find an expression for $Y(s)$, the Laplace transform of $y(t)$.

   (b) Find $y(t)$.

Table of Laplace Transforms

function $f(t)$	Laplace transform $F(s)$
1	$1/s \quad (s > 0)$
t^n	$n!/s^{n+1} \quad (s > 0)$
e^{at}	$1/(s - a) \quad (s > a)$
$\sin at$	$a/(s^2 + a^2) \quad (s > 0)$
$\cos at$	$s/(s^2 + a^2) \quad (s > 0)$
$e^{-at}f(t)$	$F(s + a)$
$\mathcal{U}(t - a)$ or $\mathcal{U}_a(t)$ $(a \geqslant 0)$	$e^{-as}/s \quad (s > 0)$
$\delta(t - a) \quad (a > 0)$	e^{-as}
$\mathcal{U}(t - a)f(t - a)$ or $\mathcal{U}_a(t)f(t - a)$	$e^{-as}F(s)$
$f^{(n)}(t)$	$s^nF(s) - s^{n-1}f(0) - s^{n-2}f'(0) \cdots - f^{(n-1)}(0)$
$f * g(t) = \int_0^t f(\tau)g(t - \tau)\, d\tau$	$F(s)G(s)$