McGILL UNIVERSITY

FACULTY OF SCIENCE

FINAL EXAMINATION

MATH 318

MATHEMATICAL LOGIC

Examiner: Professor J. Loveys
Associate Examiner: Professor M. Makkai

Date: Monday December 7, 2009.
Time: 9:00 A.M - 12:00 P.M

INSTRUCTIONS

1. Please answer questions in the exam booklets provided.
2. No calculators are permitted.
3. This is a closed book exam.
4. Use of a regular and/or translation dictionary is permitted.

This exam consists of the cover page, and 2 pages of 8 questions.

MATH 318, MATHEMATICAL LOGIC
FINAL EXAMINATION, FALL 2009

1. In each of the following case, translate the statements into propositional formulas. Decide in each case whether the set of all those formulas is consistent, justifying both your answers. Use p for "pistachios are perfect", q for "quizzes are quiet" and r for "ragweed is riotous".

   (a) (5 marks) If pistachios are perfect and quizzes are quiet, then ragweed is riotous. If pistachios are perfect, then either quizzes are quiet or ragweed is riotous. Pistachios are perfect and ragweed is not riotous.

   (b) (5 marks) If pistachios are perfect and quizzes are quiet, then ragweed is riotous. If pistachios are perfect, then either quizzes are quiet or ragweed is riotous. Pistachios are not perfect and ragweed is riotous.

2. Suppose that the binary relation R on the nonempty set A is reflexive and transitive. [We do *not* assume it is either symmetric or antisymmetric.]

   (a) (5 marks) Suppose we define E on A by aEb if and only if aRb and bRa. Show that E is an equivalence relation on A.

   (b) (5 marks) Prove that, if $a'Ea$ and $b'Eb$, then aRb if and only if $a'Rb'$.

   (c) (5 marks) As usual, we call the set of equivalence classes A/E and use $[a]$ for the class containing $a \in A$. We define a relation $\bar{R}$ of A/E by:

   $$[a]\bar{R}[b] \text{ if and only if } aRb.$$

   Prove that $\bar{R}$ is a partial order of A/E.

3. (10 marks) Suppose that $(L_1; \leq_1)$ and $(L_2; \leq_2)$ are lattices such that $L_1 \cap L_2 = \emptyset$ and $\top$ and $\bot$ are distinct things not in $L_1 \cup L_2$. We let $L = L_1 \cup L_2 \cup \{\top, \bot\}$ and define $\leq$ on L by

   $$a \leq b \text{ if and only if either } a = \bot, \text{ or } b = \top, \text{ or } a, b \in L_1 \text{ and } a \leq_1 b, \text{ or } a, b \in L_2 \text{ and } a \leq_2 b.$$

   You don't need to verify that $(L; \leq)$ is a lattice, but show that if L_1 or L_2 (or both) has at least two elements, then this lattice is not modular.

4. (20 marks, 10 for each part) One of the following statements is a logical theorem and the other is not. Identify which is which, and give a quasi-formal proof of the logical theorem; give a structure verifying that the other is not a logical theorem. f and g are unary function symbols.

   $$\forall x \forall y (fgx = fgy \to x = y) \to \forall x \forall y (fx = fy \to x = y)$$

   $$\forall x \forall y (fgx = fgy \to x = y) \to \forall x \forall y (gx = gy \to x = y)$$

5. (10 marks) We invent a structure $\mathcal{M} = (M; f^{\mathcal{M}}, P^{\mathcal{M}})$, where f is a unary function symbol and P is a binary relation symbol as follows.
$M = \{\alpha, \beta, \gamma\}$, $f^{\mathcal{M}}(\alpha) = f^{\mathcal{M}}(\beta) = \gamma$, $f^{\mathcal{M}}(\gamma) = \alpha$, and $P^{\mathcal{M}} = \{(\alpha, \alpha), (\alpha, \gamma), (\beta, \gamma), (\gamma, \gamma)\}$.
Decide whether or not

$$\mathcal{M} \models \forall x \exists y (Pxfy \wedge \neg Pfxy).$$

Justify your answer.

6. (15 marks) Give an informal proof from Peano Arithmetic of the following. You may use anything done in class or on the assignments, but not for instance the fundamental theorem of arithmetic, since we did not prove that. Recall we use the abbreviation $\pi(x)$ for "x is a prime number" and $x|y$ for "x divides y".

$$\forall x \forall y \forall z ((\pi(x) \wedge \pi(y) \wedge \pi(z) \wedge x \neq y) \rightarrow \neg((x \cdot y)|(z \cdot z \cdot z)).$$

7. (10 marks) Prove that if α is a limit ordinal, then for any ordinal β, $\beta + \alpha$ is a limit ordinal. (The addition is of course ordinal addition.)

8. (10 marks) Give a formula in prenex normal form which is equivalent to the formula

$$\forall y (Py \rightarrow Qxy) \wedge \neg \forall x (Px \wedge \exists y Qxy).$$

Here of course P is a unary predicate symbol, and Q is a binary predicate symbol.