Student Name:
Student Id#:

McGILL UNIVERSITY

FACULTY OF SCIENCE

FINAL EXAMINATION

MATH 317

NUMERICAL ANALYSIS

Examiner: Professor A.R. Humphries — Date: Thursday December 11, 2008
Associate Examiner: Professor G. Schmidt — Time: 9:00 AM - 12:00 PM

INSTRUCTIONS

1. All questions carry equal weight.
2. Answer 6 or 7 questions; credit will be given for the best 6 answers.
3. Answer questions in the exam book provided. Start each answer on a new page.
4. This is a closed book exam.
5. Notes and textbooks are not permitted.
6. Non-programmable calculators are permitted.
7. Translation dictionaries (English-French) are permitted.

This exam comprises of the cover page, and 3 pages of 7 questions.

1. (a) State the "Fixed Point Theorem," which gives sufficient conditions for an iteration $x_{n+1} = g(x_n)$ to converge to a fixed point.

   (b) Consider the iteration with $g(x) = x + \frac{1}{2}(2 - e^x)$.

      i. Show that the iteration has a fixed point at $x = \ln 2$.

      ii. Show that the scheme satisfies all the conditions of the fixed point theorem on the interval $[0, 1]$.

      iii. What is the order of convergence of the scheme?

      iv. Let $x_0 = 0.5$ and compute x_3.

      v. What is the relative error of x_3 as an approximation to $\ln 2$?

2. Assume that $x_0 < x_1 < \ldots < x_n$. Then divided differences can be defined recursively using the formula

$$f[x_i, x_{i+1}, \ldots, x_{i+j}] = \frac{f[x_{i+1}, x_{i+2}, \ldots, x_{i+j}] - f[x_i, x_{i+1}, \ldots, x_{i+j-1}]}{x_{i+j} - x_i}.$$

   (a) Define the zeroth divided differences $f[x_j]$ for $j = 0, 1, \ldots, n$.

   (b) Let

$$p_n(x) = \sum_{j=0}^{n} c_j w_j(x)$$

   be the Newton form of the interpolating polynomial based on $x_0, x_1, \ldots, x_n$. Define the polynomials $w_j(x)$ for $j = 0, 1, \ldots, n$. State c_j in terms of the appropriate divided difference(s) for $j = 0, 1, \ldots, n$.

   (c) Assuming that f is n-times differentiable, show that there exists $\xi \in [x_0, x_n]$ such that

$$f[x_0, x_1, \ldots, x_n] = \frac{f^{(n)}(\xi)}{n!}.$$

   (d) Given

$$x_0 = 0, \quad x_1 = 1, \quad x_2 = 2, \quad x_3 = 3,$$

$$f(x_0) = 2, \quad f(x_1) = 3, \quad f(x_2) = 10, \quad f(x_3) = 29,$$

   construct the appropriate table of divided differences and hence state

      i. the polynomial of degree 3 which interpolates at x_0, x_1, x_2, x_3.

      ii. the polynomial of degree 2 which interpolates at x_1, x_2, x_3.

   Evaluate each polynomial at $x = 2.5$.

3. Consider the Forward Difference Approximation

$$f'(x_0) \approx N_1(h) = \frac{f(x_0 + h) - f(x_0)}{h},$$

and the data

x	0	0.1	0.2	0.4
$f(x)$	0	0.099833	0.19867	0.38942

(a) Using Taylor Series, or otherwise, show that $f'(x_0) = N_1(h) + c_1 h + c_2 h^2 + \mathcal{O}(h^3)$.

(b) Use Richardson extrapolation to find $N_2(h)$ such that $f'(x_0) = N_2(h) + k_2 h^2 + \mathcal{O}(h^3)$, and $N_3(h)$ such that $f'(x_0) = N_3(h) + \mathcal{O}(h^3)$. (The formula for $N_2(h)$ should involve $N_1(h)$ and $N_1(h/2)$.

(c) Taking $x_0 = 0$, evaluate $N_1(0.1)$, $N_1(0.2)$ and $N_1(0.4)$, and use these values to evaluate $N_2(h)$ for two values of h and $N_3(h)$ for one value of h.

4. (a) Define the degree of accuracy (also known as the degree of precision) of a quadrature formula $I_h(f)$ for approximating the integral

$$I(f) = \int_a^b f(x)dx.$$

(b) Find the degree of accuracy p of the quadrature formula

$$I_h(f) = \frac{3}{2}h[f(x_1) + f(x_2)]$$

where $a = x_0$, $b = x_3$ and $h = x_{i+1} - x_i$.

(c) Given that $I(f) = I_h(f) + kh^{p+2}f^{(p+1)}(\xi)$, where p is the degree of accuracy, find k.

(d) Evaluate $I_h(f)$ when $I(f) = \int_1^2 \frac{1}{x}dx = \ln(2)$ to obtain an approximation to $\ln(2)$. Use the error bound from (c) to find an upper bound for the error in this approximation.

5. Simpson's Rule $J_h(f) = (h/3)[f_0 + 4f_1 + f_2]$, where $f_i = f(x_i)$ for approximating $J(f) = \int_{x_0}^{x_2} f(x)dx$ has the error formula

$$J(f) - J_h(f) = -\frac{h^5}{90} f^{(4)}(\zeta)$$

where $\zeta \in [x_0, x_2]$.

(a) Let n be even, $x_0 = a$, $x_n = b$, $h = (b-a)/n$ and $x_j = a + jh$. State the Composite Simpson's Rule $I_h(f)$ for approximating $I(f) = \int_a^b f(x)dx$.

(b) Assuming that $f \in C^4[a,b]$ show that the Composite Simpson's Rule satisfies

$$I(f) - I_h(f) = -\frac{(b-a)}{180} h^4 f^{(4)}(\xi)$$

for some $\xi \in [a,b]$.

(c) Let $I_h(f)$ be the Composite Simpson's Rule approximation to

$$\ln 2 = I(f) = \int_1^2 \frac{1}{x} dx.$$

i. Evaluate $I_h(f)$ with $h = 0.25$.

ii. What value of h is required to ensure that $|I(f) - I_h(f)| \leqslant 10^{-8}$?

6. Consider the initial value problem

$$y' = f(y), \qquad 0 \leqslant t \leqslant T, \quad y(0) = \alpha.$$

Suppose you approximate the solution $y(t)$ using the Runge-Kutta method

$$w_0 = \alpha,$$
$$w_{i+1} = w_i + \tfrac{1}{3}h\Big[f(w_i) + 2f(w_i + \tfrac{3h}{4}f(w_i))\Big], \quad i = 0, \ldots N$$

with time-step $h > 0$.

(a) Define and find the local truncation error $\tau_{i+1}(h)$ of this method, and use it to determine the order of the method.

(b) Consider the case where

$$f(y) = \lambda y, \quad \lambda < 0,$$

and

i. show that $w_{i+1} = (1 + h\lambda + \frac{(h\lambda)^2}{2})w_i$.

ii. Under what conditions on h does $\lim_{i\to\infty} w_i = 0$?

7. (a) State sufficient conditions on $p(t)$, $q(t)$, $r(t)$, to ensure that the boundary value problem

$$y'' = p(t)y' + q(t)y + r(t), \qquad a \leqslant t \leqslant b, \qquad y(a) = \alpha, \quad y(b) = \beta,$$

has a unique solution.

(b) Use the linear shooting method to approximate the solution $y(0.5)$ of the boundary value problem

$$y'' = ty' + 2y - t, \qquad 0 \leqslant t \leqslant 1, \qquad y(0) = 0, \quad y(1) = 2,$$

using $h = \frac{1}{2}$, and the (Forward) Euler method.