MATH 251, HONOURS ALGEBRA 2, FINAL EXAMINATION, APRIL 30, 2010

On this exam $\mathcal{R}$ denotes the field of real numbers and $\mathcal{C}$ the field of complex numbers.
PART I. (Each of these problems is worth 6 marks.)

1. Let $V = M_2(\mathcal{R})$, the real vector space of 2×2 matrices over $\mathcal{R}$. Let $T : V \longrightarrow V$ be defined by

$$TX = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} X - X \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.$$

   (a) (2 marks) Verify that T is a linear operator on V.

   (b) (4 marks) Find a basis for each of $ker(T)$ and $im(T)$.

2. (6 marks) Find, explicitly, $\begin{pmatrix} 3 & 0 & -1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix}^{15}$. [N.B. $2^{15} = 32768$.]

3. (6 marks) Find the general solution to the following system of differential equations:

$$\begin{cases} y_1' &= y_1 + 6y_2 \\ y_2' &= 4y_1 + 3y_2 \end{cases}$$

4. (6 marks) The real matrix $\begin{pmatrix} 0 & 0 & 0 & 5 & 0 & 0 \\ 1 & 0 & 0 & 14 & 0 & 0 \\ 0 & 1 & 0 & 12 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & -2 \end{pmatrix}$ is in rational canonical form. Find its Jordan form.

5. (6 marks) Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $B = \begin{pmatrix} a & b \\ \widehat{c} & \widehat{d} \end{pmatrix}$ be matrices with entries from $\mathcal{Z}_7$, and $det(A) = 3$, $det(B) = 2$. If $C = \begin{pmatrix} 5a & 4c + \widehat{c} + 3a \\ 5b & 4d + \widehat{d} + 3b \end{pmatrix}$, is C invertible? Justify. (Recall: $\mathcal{Z}_7$ is the field with 7 elements.)

6. (6 marks) $B = \left(\begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix} \right)$ is an ordered basis for $\mathcal{R}^4$. Find the value of $f_j(\vec{e}_k)$ for $1 \leq j, k \leq 4$, where (f_1, f_2, f_3, f_4) is the dual basis B^* of $(\mathcal{R}^4)^*$.

7. (6 marks) Find a unitary matrix U such that $\overline{U}^T HU$ is diagonal, and find the diagonal matrix, where $H = \begin{pmatrix} 1 & 1+i & 1-i \\ 1-i & 2 & -2i \\ 1+i & 2i & 2 \end{pmatrix}$.

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PART II (Each of these problems is worth a total of 12 marks.)

1. Recall that $M_{m,n}(F)$ is our notation for the vector space of $m \times n$ matrices over the field F. Suppose that $\vec{a} \in F^n$ is nonzero and $V = M_{m,n}(F)$.
   (a) (3 marks) Show that $\{X \in V : X\vec{a} = \vec{0}\}$ is a subspace of V.
   (b) (9 marks) What is the dimension of the subspace in part (a)? Justify your answer.

2. (a) (8 marks) Suppose that U and W are subspaces of the vector space V. Prove that $U/(U \cap W)$ is isomorphic to $(U+W)/W$.
   (b) (4 marks) In case U and W are finite-dimensional, use this to give a second proof that
$$dim(U) + dim(W) = dim(U+W) + dim(U \cap W).$$

3. (12 marks) Suppose that T is a linear operator on the vector space V over F, $\vec{v}$ is an eigenvector for T, and that $p(x) \in F[x]$. Show that $\vec{v}$ is also an eigenvector for $p(T)$.

4. (12 marks) Suppose that A is an $n \times n$ matrix over F and that the minimal polynomial min_A splits over F. Show that there are matrices B and C over F such that:
   (a) $A = B + C$;
   (b) B is diagonalizable; and
   (c) $C^k = 0$ for some natural number k.

5. (12 marks) Suppose that V is a vector space over either $\mathcal{R}$ or $\mathcal{C}$, and B is a basis for V. Show that there is exactly one way to define an inner product on V such that B is an orthonormal basis of V with respect to this inner product.

6. The set V of all sequences of real numbers $(x_1, x_2, \ldots, x_n, \ldots)$ such that $\sum_{n=1}^{\infty} x_n^2$ is finite is a real vector space. You may assume this for this problem.
   (a) (10 marks) Show that for any $(x_1, x_2, \ldots, x_n, \ldots) \in V$,
$$\left(\sum_{n=1}^{\infty} \frac{x_n}{2^n}\right)^2 \leq \frac{1}{3} \sum_{n=1}^{\infty} x_n^2.$$
   (b) (2 marks) Identify the sequences in V for which we have equality in the last part; justify.

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PART III. (Each of the following questions is worth a total of 20 marks. A full exam will contain an attempt at at least two of these; you cannot receive more than 60% from problems taken from PART I and PART II.)

1. (20 marks) Recall that GF_{p^k} is the finite field with p^k elements, where p is a prime. Show that GF_{p^k} has a subfield isomorphic to GF_{p^j} if and only if $j|k$.

2. (20 marks) Show that, if A is any square matrix with entries from the field F, then A is similar to A^T.

   [I can think of two ways to do this, one going through the Jordan form, the other using one of the exercises related to the rational form.]

3. (20 marks) Suppose that V is an infinite-dimensional vector space over F, and $T : V \longrightarrow V$ is linear. Suppose also that W is a T-invariant subspace of V. Show that there is a subspace U of V with the following three properties:

   (a) U is T-invariant;

   (b) $U \cap W = \{\vec{0}\}$; and

   (c) for any $\vec{v} \in V$, there is a polynomial $p(x) \in F[x]$ such that $p(T)\vec{v} \in U + W$.

4. (20 marks) Suppose that V is inner product space and W_1 and W_2 are subspaces of V.

   (a) (10 marks) Show that

   $$(W_1 + W_2)^\perp = W_1^\perp \cap W_2^\perp \text{ and } W_1^\perp + W_2^\perp \subseteq (W_1 \cap W_2)^\perp.$$

   (b) (10 marks) Show that, if V is finite-dimensional, then

   $$W_1^\perp + W_2^\perp = (W_1 \cap W_2)^\perp.$$