MATH 251, HONOURS ALGEBRA 2, FINAL EXAMINATION, APRIL 24, 2009

On this exam $\mathcal{R}$ denotes the field of real numbers, $\mathcal{C}$ the field of complex numbers, and $\mathcal{Z}_2$ the 2-element field.

PART I. (Each of these problems is worth 6 marks.)

1. Let $V = P_2(X)$, the real vector space of polynomials of degree at most 2. Let $T : V \longrightarrow V$ be defined by
$$Tp(X) = (X^2 + 2)p''(X) - 2p(X).$$
   (a) (2 marks) Verify that T is a linear operator on V.
   (b) (4 marks) Find a basis for each of $ker(T)$ and $im(T)$.

2. Give an explicit, nonrecursive, formula for x_n, where x_n is defined recursively by
$$x_0 = x_1 = 3, \quad x_{n+2} = 6x_{n+1} - 9x_n \text{ for } n \geq 0.$$

3. Suppose that V is a 10-dimensional vector space over $\mathcal{R}$ and T is a linear operator on V with eigenvalues 1 and -2. Suppose also that $dim(ker(T - I)) = 3$, $dim(ker(T-I)^2) = 5$, $dim(ker(T-I)^3) = dim(ker(T-I)^4) = 7$, $dim(ker(T+2I)) = 2$ and $dim(ker(T+2I)^2) = dim(ker(T+2I)^3) = 3$. Write down the Jordan form of T.

4. Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $B = \begin{pmatrix} a & b \\ \hat{c} & \hat{d} \end{pmatrix}$ be matrices with complex entries, $det(A) = 3 - i$ and $det(B) = 2i$. If $C = \begin{pmatrix} (1+i)a & 2c + 2i\hat{c} + a \\ (1+i)b & 2d + 2i\hat{d} + b \end{pmatrix}$, what is $det(C)$? Justify.

5. Let
$$B = \left(\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \right)$$
be an ordered basis for $\mathcal{R}^3$. Find the value of $f_j(\vec{e}_k)$ for $1 \leq j, k \leq 3$, where (f_1, f_2, f_3) is the dual basis B^* for $(\mathcal{R}^3)^*$.

6. Find an orthonormal basis for W and an orthonormal basis for $W^\perp$, where W is the following subspace of $\mathcal{R}^3$.
$$W = Span \left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 4 \\ 1 \\ 2 \end{pmatrix} \right\}.$$

7. Find a unitary matrix U such that $\overline{U}^T HU$ is diagonal, and find the diagonal matrix, where $H = \begin{pmatrix} 3 & 1-i \\ i+1 & 2 \end{pmatrix}$.

PART II (Each of these problems is worth a total of 12 marks.)

1. (12 marks) Suppose that U and W are subspaces of the vector space V and $U \leq W$. Prove that $(V/U)/(W/U)$ is isomorphic to V/W.

2. (a) (6 marks) Suppose that A is a diagonalizable matrix with real entries. Show that there is a matrix C with real entries such that $C^3 = A$.

   (b) (6 marks) Find a matrix B with real entries such that $B^3 = \begin{pmatrix} 17 & 9 \\ -18 & -10 \end{pmatrix}$.

3. For any particular matrix $A \in M_n(F)$, we let $Z(A) = \{X \in M_n(F) : XA = AX\}$.

   (a) (3 marks) Show that $Z(A)$ is a subspace of $M_n(F)$.

   (b) (4 marks) Show that, if A and B are similar over F, then $Z(A)$ and $Z(B)$ have the same dimension over F.

   (c) (5 marks) If $n = 2$ and $F = \mathcal{C}$, prove that the dimension of $Z(A)$ over $\mathcal{C}$ must be 2 or 4.

4. Let $A = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix}$ be a matrix with entries from $\mathcal{Z}_2$.

   (a) (3 marks) Find the minimal polynomial min_A of this matrix.

   (b) (9 marks) With the operations being matrix addition and multiplication, show that the set F below is a field and determine how many elements it has.
   $$F = \{p(A) : p(X) \in \mathcal{Z}_2[X]\}$$

5. Let V be the vector space of continuous functions on $[0, \pi]$ and
   $$\langle f, g \rangle = \int_0^\pi f(x)g(x)\sin x dx \text{ for } f, g \in V.$$

   (a) (3 marks) Verify that this defines an inner product on V.

   (b) (9 marks) Show that, for any $f \in V$,
   $$\left(\int_0^\pi f(x) \sin x \cos x dx \right)^2 \leq \frac{2}{3} \int_0^\pi f(x)^2 \sin x dx$$
   and identify those functions $f \in V$ for which equality holds.

6. (12 marks) Suppose that V is a finite-dimensional inner product space and $V = W_1 \oplus W_2$, where W_1 and W_2 are subspaces of V. Show that $V = W_1^\perp \oplus W_2^\perp$.

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PART III. (Each of the following questions is worth a total of 20 marks. A full exam will contain an attempt at at least two of these; you cannot receive more than 60% from problems taken from PART I and PART II.)

1. Recall that the operator T on the vector space V is called nilpotent if $T^k = 0$ for some natural number k.
   (a) (12 marks) Show that, if T_1 and T_2 are nilpotent operators on V and $T_1T_2 = T_2T_1$, then T_1T_2 and $T_1 + T_2$ are also nilpotent.
   (b) (8 marks) Give an example of operators T_1 and T_2 on a space V such that T_1 and T_2 are both nilpotent, but T_1T_2 and $T_1 + T_2$ are not. (Of course, you can't have $T_1T_2 = T_2T_1$. You may use operators that come from matrices.)
2. Suppose that T_1 and T_2 are linear operators on the vector space V such that $T_1T_2 = T_2T_1$.
   (a) (8 marks) Show that, for any λ and k, $ker(T_1 - \lambda I)^k$ is T_2-invariant.
   (b) (12 marks) If we further suppose that V is finite-dimensional and each of T_1 and T_2 is diagonalizable, then there is an ordered basis B such that both $[T_1]_B$ and $[T_2]_B$ are diagonal.
3. (20 marks) Suppose that F is a field with finitely many elements. Show that the multiplicative group of F is cyclic. [Hint: Use the main result from the posted notes on abelian groups.]
4. (20 marks) Suppose that V is a vector space over either $\mathcal{R}$ or $\mathcal{C}$, W is a subspace of V, and we are given an inner product on W. Show that there is at least one way to extend this function to an inner product on all of V. Do not assume that V is finite-dimensional.