MCGILL UNIVERSITY
FACULTY OF SCIENCE

Final Examination

MATH 248
HONOURS ADVANCED CALCULUS

Examiner: Professor V. Jaksic
Associate Examiner: Professor G. Schmidt

Thursday December 6, 2007
Time: 9:00 AM to 12:00 PM

Family Name (Please Print): ______________________

First Name: ________________________

Student ID#: ____________________

INSTRUCTIONS

1. Fill in the above clearly.
2. Do not tear any pages from this book.
3. Write your solutions in a clear, complete and logical way.
4. There are 7 questions worth a total of 80 points. The value of each question is indicated in the margin.
5. This is a closed book examination. No, notes, books or calculators are allowed.
6. Use of a regular and or translation dictionary is not permitted.
7. This examination consists 18 pages including this cover page. There are four empty pages at the end of this exam. You may use them if you need extra space.

1. Let $f : \mathbf{R} \mapsto \mathbf{R}$ be a C^1 function and let

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$$u(x, y) = xyf\left(\frac{x+y}{xy}\right).$$

Show that u satisfies equation of the form

$$x^2 \frac{\partial u}{\partial x} - y^2 \frac{\partial u}{\partial y} = G(x, y)u,$$

and find $G(x, y)$.

2. Let D be the solid cylinder $x^2 + y^2 \leq a^2, 0 \leq z \leq h$. Compute

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$$\iiint_D \frac{z - R}{\left(x^2 + y^2 + (z - R)^2\right)^{3/2}} dxdydz$$

where $R < 0$.

Additional page for Problem 2.

3. Let

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$$D = \left\{ (x, y) \,:\, \frac{x^2}{a^2} + \frac{y^2}{b^2} \leq 1 \right\}$$

and let $f, g : D \mapsto \mathbf{R}$ be two C^1 functions. Define two vector fields F and G by

$$F(x, y) = g(x, y)\mathbf{i} + f(x, y)\mathbf{j}, \qquad G(x, y) = \left(\frac{\partial f}{\partial x} + 2\frac{\partial f}{\partial y} \right) \mathbf{i} + \left(\frac{\partial g}{\partial x} + 2\frac{\partial g}{\partial y} \right) \mathbf{j}.$$

Find the value of the double integral

$$\iint_D F \cdot G \, dx dy,$$

if it is known that on the boundary of D we have $f(x, y) = x$ and $g(x, y) = 1$.

Additional page for Problem 3.

4. Using Stokes' theorem compute

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$$\oint_\gamma (y^2 - z^2)dx + (z^2 - x^2)dy + (x^2 - y^2)dz$$

where γ is the curve cut from the boundary of the cube $0 \leq x \leq a$, $0 \leq y \leq a$, $0 \leq z \leq a$ by the plane $x + y + z = 3a/2$. The path is traversed in a direction that appears counterclockwise when viewed from high above the xy-plane.

Additional page for Problem 4.

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5. The cylinder $x^2 + y^2 = 2x$ cuts out a portion of a surface S from the cone $z = \sqrt{x^2 + y^2}$. Compute the value of the surface integral

$$\iint_S (xy + yz + zx)dS.$$

Additional page for problem 5.

6. Let S be the surface described in the Problem 5 and let γ be the boundary of S oriented counterclockwise when viewed high above the yz-plane. Let

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$$I = \oint_\gamma (y - z)dx + (z - x)dy + (x - y)dz.$$

(a) [10 points] Compute I using an appropriate parametrization of γ.

(b) [10 points] Compute I using Stokes' theorem.

Additional page for problem 6.

7. Using the divergence theorem evaluate the surface integral

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$$\iint_S \mathbf{F} \cdot \mathbf{n} dS,$$

where S is the surface

$$|x - y + z| + |y - z + x| + |z - x + y| = 1,$$

$\mathbf{n}$ is the unit outer normal to S, and

$$\mathbf{F}(x, y, z) = (x - y + z)\mathbf{i} + (y - z + x)\mathbf{j} + (z - x + y)\mathbf{k}.$$

Additional page for Problem 7.

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