Winter 2011 — Mathematics 243 (Analysis 2)

Final Exam

Monday, April 11, 2011, 2:00pm.

Instructions

(i) No books, no calculators.
(ii) Work on all 6 problems.
(iii) Remember: $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$, $\lim_{n\to\infty}(1+\frac{1}{n})^n = e$

Definitions

(F1) Let (a_n) be a sequence of real numbers.

(a) Define what it means for the series $\sum a_n$ to be convergent.
(b) Define what it means for the series $\sum a_n$ to be absolutely convergent.
(c) How does one show that if $\sum a_n$ is absolutely convergent, then it is convergent? (Briefly state the main idea.)
(d) Give an example of a series that is convergent, but not absolutely convergent.

Substitution

(F2) For $n \in \mathbb{N}_0$, consider the integral

$$c_n = \int_0^\infty x^n\, e^{-x^2} dx$$

(a) Show that for $n > 1$, $c_n = \frac{n-1}{2} c_{n-2}$.
(b) Evaluate c_1, and c_{2m+1} for $m \geq 1$.
Hint: You may integrate by parts, or first substitute $x^2 = y$, and then integrate by parts.

Integration

(F3) Let $\lfloor \cdot \rfloor : \mathbb{R} \to \mathbb{Z}$ be the function

$$x \mapsto \lfloor x \rfloor := n \qquad \text{for the unique } n \in \mathbb{Z} \text{ with } x \in [n, n+1).$$

(i.e., $\lfloor x \rfloor$ is the largest integer not larger than x). Show that the integral

$$\int_0^1 \frac{1}{\lfloor \frac{1}{x} \rfloor} dx$$

exists, and compute its value.

Please turn over

Sequence of Functions

(F4) Consider the sequence of functions (f_n) on $\mathbb{R}$ defined by

$$f_n(x) = (\cos x)^n$$

(a) On which subset of $\mathbb{R}$ does (f_n) converge, on which does it not?
(b) Find an open interval on which (f_n) converges uniformly.

Weak Convergence

(F5) Recall that a function $F : [0, \infty) \to \mathbb{R}$ is called integrable on $[0, \infty)$ if it is integrable on $[0, x]$ for any $x > 0$ and if $\lim_{x\to\infty} \int_0^x F$ exists.
Consider the sequence of functions (g_n) on $\mathbb{R}$ defined by

$$g_n(x) = \begin{cases} 1 & \text{for } x \in [n, n+1] \\ 0 & \text{else} \end{cases}$$

Show that

(a) (g_n) converges pointwise on $[0, \infty)$.
(b) (g_n) does not converge uniformly on $\mathbb{R}$.
(c) For any $n \in \mathbb{N}$, g_n is integrable on $[0, \infty)$, but $\lim_{n\to\infty} \int_0^\infty g_n \neq \int_0^\infty \lim_{n\to\infty} g_n$.
(d) If $f : [0, \infty) \to \mathbb{R}$ is integrable on $[0, \infty)$, then

$$\lim_{n\to\infty} \int_0^\infty f \cdot g_n = 0$$

Power Series

(F6) Determine the radius of convergence of the power series

$$\sum_{n=1}^{\infty} \frac{n^n}{n!} x^n$$