1i(6 marks) The *Archimedean property* states that the set of natural numbers $\mathbb{N}$ is unbounded in $\mathbb{R}$. Assuming this property, show that $\inf\{1/n : n \in \mathbb{N}\} = 0$. Conclude that if $t > 0$, there exists some $n \in \mathbb{N}$ with $\frac{1}{n} < t$, and if $y > 0$, there exists a natural number n such that $n - 1 \leq y < n$.

1ii(6 marks) Use **1i** to show that for any two real numbers x and y with $x < y$, there exists a rational number r with $x < r < y$.

2i(3 marks) Let $f : A \to \mathbb{R}$ be a function. Define what it means for f to be *continuous* on A, and *uniformly continuous* on A. Comment on the difference between these definitions.

2ii (4 marks) Show that if f and g are uniformly continuous functions on A, and they are both bounded on A, then their product fg is uniformly continuous on A.

2iii (4 marks) Show that if A is the closed bounded interval $A = [a, b]$, then a continuous function $f : A \to \mathbb{R}$ must be bounded. State any theorems you use. Conclude that the product of two uniformly continuous functions defined on a closed bounded interval is uniformly continuous.

2iv(3 marks) Give an example of a set A and a uniformly continuous function $f : A \to \mathbf{R}$ which is not bounded, justifying your choice.

3i(4 marks) Let $A \subset \mathbb{R}$ be a nonempty set. Define what it means for a real number to be the *supremum* of A. State the *completeness axiom* for $\mathbb{R}$.
Now let $I_n = [a_n, b_n]$, $n \in \mathbb{N}$ be a *nested* sequence of intervals, i.e. $I_{n+1} \subset I_n$for each $n \in \mathbb{N}$.

3ii(2 marks) Show that the set $\{a_n : n \in \mathbb{N}\}$ is bounded above, and let a^* be its supremum.

3iii(5 marks) Show that $a^* \in \cap_{n=1}^{\infty} I_n$.

3iv (6 marks) If $f : [a, b] \to \mathbb{R}$ is a continuous function, with $f(a) < 0$ and $f(b) > 0$, show that there exists $x \in (a, b)$ with $f(x) = 0$ (Hint: Use **3iii**).

3v(3 marks) Show that the polynomial $f(x) = x^4 + 7x^3 - 9$ has at least 2 real roots. Describe how you would locate one of these roots x with error less than 0.005.

4(12 marks) Prove or disprove each of the following statements:

4i If every subsequence of (x_n) has a subsequence that converges to 0, then $\lim_{n\to\infty} x_n = 0$.

4ii The sequence $(1 + (-1)^n)$ is a Cauchy sequence.

4iii The sequence $((3n)^{\frac{1}{2n}})$ diverges to ∞.

5i(5 marks) If $0 < x < 1$, show that $x^n \to 0$ as $n \to \infty$.

5ii(3 marks) If $0 < a < b$, find the limit of the sequence $(\frac{a^{n+1}+b^{n+1}}{a^n+b^n})$, *stating* any theorems that you use.

6i(6 marks) Suppose $f : [a, b] \to \mathbb{R}$ with $c \in (a, b)$. Show that f is differentiable at c if and only if there exists a function $\phi : [a, b] \to \mathbb{R}$ such that ϕ is continuous at c and satisfies

$$f(x) - f(c) = \phi(x)(x - c)$$

whenever $x \in [a, b]$. Show that in this case, $f'(c) = \phi(c)$.

6ii(3 marks) If $f(x) = x^3$, find $\phi(x)$ and deduce that $f'(c) = 3c^2$.

6iii(7 marks) Suppose f is strictly monotone increasing and continuous on $[a, b]$. Let $J = f([a, b])$ and consider the function $g : J \to [a, b]$ defined as $g = f^{-1}$. Explain why J is a closed bounded interval. Show that if f is differentiable at $c \in [a, b]$ and $f'(c) \neq 0$, then f^{-1} is differentiable at $d := f(c)$ and $g'(d) = \frac{1}{f'(c)}$ (Hint: use **6i**). Using **6ii**, if $f(x) = x^3$, find $g'(d)$.

7 (8 marks) Let f, g be differentiable on $\mathbb{R}$ and suppose that $f(0) = g(0)$ and $f'(x) \leq g'(x)$ for all $x \geq 0$. Show that $f(x) \leq g(x)$ for all $x \geq 0$. State completely any theorems that you use.