Prof. Vermes

(15)

Final Examination December 19, 2002 MATH 242

1. (a) Define:
   (i) Least upper bound of a bounded set $S \subset \mathbb{R}$;

   (ii) $X = (x_k), x_k \in \mathbb{R}$, is a convergent sequence;

   (iii) $X = (x_k), x_k \in \mathbb{R}$ is a Cauchy sequence;

   (iv) A function f defined on $S \subset \mathbb{R}$ is uniformly continuous on S.

   (b) State the Least Upper Bound Axiom.

2. Let $A \subset \mathbb{R}, B \subset \mathbb{R}$ be two non-empty bounded sets.
   Show that the set
   $$C = \{c : c = a + b, \quad a \in A, \quad b \in B\}$$
   is bounded and Sup C = Sup A + Sup B.

3. (a) Let $a > 0$, prove that
   $$\lim_{n\to\infty} \sqrt[n]{a} = 1.$$

   (b) Let $a_1, a_2, \ldots, a_{k-1}, a_k$ be k positive numbers.
   Prove that
   $$\lim_{n\to\infty} (a_1^n + a_2^n + \cdots + a_{k-1}^n + a_k^n)^{\frac{1}{n}} = \max(a_1, a_2, \ldots, a_k).$$

4. (a) Show that every increasing bounded sequence (a_k) is convergent.

   (b) Let f be defined and increasing on the interval (a,b). Prove that for all $c \in (a, b)$ we have that $\lim\limits_{x\to c^+} f$ and $\lim\limits_{x\to c^-} f$ exist.

5. (a) Let f be defined on the punctured neighborhood $N = \{x : 0 < |x - a| < \lambda\}$.
   If $\lim\limits_{x\to a} f(x) =$ A, prove that f is bounded on a punctured neighborhood $\{x : 0 < |x - a| < \beta \le \lambda\}$.

   (b) Let g be defined and positive on $\{x : 0 < |x| < \lambda\}$.
   Suppose that $\lim\limits_{x\to 0} (g(x) + \frac{1}{g(x)})$ =2.

   Prove that $\lim\limits_{x\to 0} g(x)$ exists and it is equal to 1.

6. (a) State and prove the Intermediate Value Theorem.

   (b) Let f be defined and continuous on [0,1]. If $0 \leq f \leq 1$, show that there exists a $\zeta \in [0,1]$ such that $f(\zeta) = \zeta$.

7. (a) State Rolle's Theorem.

   (b) Prove the Mean Value Theorem.

   (c) Let f be continuous on $[0,1]$ and differentiable on $(0,1)$. Suppose that $f(0) = f(1) = 0$ and that there is an $x_0 \epsilon (0,1)$ such that $f(x_0) = 1$. Prove that $|f'(c)| \geq 2$ for some $c \epsilon (0,1)$.

   (d) (This question is for extra points.) Show that in (c) the strict inequality $|f'(c)| > 2$ is true.

8. (a) State Taylor's Theorem, with the Lagrange remainder.

   (b) Establish the inequality

$$1 + rx + \frac{r(r-1)}{2}x^2 \leq (1+x)^r$$

   if $x \geq 0$ and r$\geq$ 2.