McGILL UNIVERSITY

FACULTY OF SCIENCE

FINAL EXAMINATION

MATH 240

DISCRETE STRUCTURES 1

Examiner: Professor A. Vetta
Associate Examiner: Professor B. Shepherd

Date:Monday December 10, 2007
Time: 9:00 AM- 12:00 PM

INSTRUCTIONS

1. Please answer questions in the exam booklets provided. Write your answer clearly.
2. This is a closed book exam.
3. Calculators are not permitted.
4. Regular and or translation dictionaries are not permitted.

This exam comprises of the cover page and two pages of six questions.

Math 240: Discrete Structures I *Monday 10th December*

Final Exam

Instructions. The exam is 3 hours long and contains 6 questions. Write your answers clearly in the notebook provided. You may quote any result/theorem seen in the lectures or in the assignments without proving it (unless, of course, it is what the question asks you to prove).

1. *Logic.*

(a) Give the negation of the statement

$$\forall n \in \mathbb{N} \;\; (n^3 + 6n + 5 \text{ is odd } \Rightarrow n \text{ is even})$$

(b) Either the original statement in a) or its negation is true. Which one is it?

(c) Use a truth table to decide whether or not the following implication is a tautology:

$$(p \wedge (p \Rightarrow q)) \;\Rightarrow\; q$$

(d) Prove that $\overline{p} \Rightarrow (q \Rightarrow r)$ and $q \Rightarrow (p \vee r)$ are equivalent.

2. *Number Theory.*

(a) State Fermat's Little Theorem.

(b) Show that Fermat's Little Theorem does not hold if p is not prime.

(c) Evaluate 302^{302} mod 11.

3. *Combinatorics.*

(a) Consider the set $[n] = \{1, 2, \ldots, n\}$. How many subsets does it have of cardinality k and that contain the element 1?

(b) Prove, by algebraic manipulation, that

$$\binom{2n}{n} + \binom{2n}{n+1} = \frac{1}{2}\binom{2n+2}{n+1}$$

(c) Prove b) using a combinatorial argument.

4. *Recurrences.*

   Let $f(n)$ be the number of ways to pay *exactly* \$$n$ if we are only allowed to use \$5 and \$10 bills.

   (a) Give a recurrence relation for $f(n)$.

   (b) Solve the recurrence equation.

5. *Graph Theory.*

   (a) State, without proof, necessary and sufficient conditions for an undirected connected graph G to contain an Euler circuit.

   (b) Take a graph G on $n \geq 4$ vertices and suppose that every vertex has degree at least $\lfloor \frac{1}{2}n \rfloor$. Does G necessarily contain a Hamiltonian cycle? (Either give a proof or provide a counter-example.)

   (c) Does there exist a planar graph whose edges can be coloured either red, green or blue in such a way that the red edges form a spanning tree, the green edges form a spanning tree, and the blue edges form a spanning tree?

6. *Trees.*

   (a) How many labelled (spanning) trees are there on n vertices?

   (b) Given the Prüfer code $(3, 3, 4, 6, 0, 1)$, what is the corresponding unlabelled tree?

   (c) An edge e in a connected graph $G = (V, E)$ is *critical* if $G' = (V, E - \{e\})$ is not connected. Prove that a connected graph is a tree if and only if every edge is critical.