

Name: _____

Student number: _____

VERSION A

McGILL UNIVERSITY
FACULTY OF SCIENCE

FINAL EXAMINATION

MATH 235

ALGEBRA 1

Examiner: Professor E. Goren
Associate Examiner: Professor J. Loveys

Date: Wednesday December 9, 2009
Time: 2:00 PM - 5:00 PM

INSTRUCTIONS

1. This exam has TWO PARTS. Please answer the questions in **PART A** of this exam directly on **the exam sheet** and return it to the invigilator. Please answer the questions in **PART B** of this exam in the exam booklet and return to the invigilator.
2. Follow the instructions given in each section of this exam.
3. Write your answers clearly.
4. This is a closed book exam. Notes or books are not permitted.
5. Calculators are not permitted.
6. Use of a regular and/or translation dictionary is not permitted.

This exam comprises the cover page, and 3 pages of questions.

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Version A

Instructions. The use of books, notes, dictionaries and any other material is not allowed. The use of calculators is not allowed. Write your answers clearly following the instructions. Points will be deducted for messy, or ambiguous, solutions. The questionnaire must be returned.

Notation. For a positive integer n we denote by $\mathbb{Z}/n\mathbb{Z}$ the ring of congruence classes modulo n . For a prime p , we use \mathbb{F}_p to denote a field with p elements, which you can take to mean $\mathbb{Z}/p\mathbb{Z}$. For a positive integer n , we let S_n denote the symmetric group on n symbols; it is a group with $n!$ elements. We let D_n denote the dihedral group with $2n$ elements - the group of symmetries of a regular n -gon in the plane. We denote the cardinality of a set A by either $|A|$ or $\# A$.

Part I. Multiple choice questions. 36 points. Each question has a unique correct answer. Each question is worth 4 points. There is no penalty for not answering. If you have more than one wrong answer then each additional wrong answer carries a penalty of 2 points. Write the answers in the table provided here.

1	2	3	4	5	6	7	8	9

- (1) The group S_{17} has elements of order:
 - (a) 3 and 66.
 - (b) 19 and 66.
 - (c) 3 and 19.
- (2) Let G be a group and H a subgroup of G . Define a relation on elements of G by saying that $a \sim b$ if $b^{-1}a \in H$. This relation is:
 - (a) reflexive and transitive, but symmetric only if G is abelian.
 - (b) reflexive and symmetric, but transitive only if G is abelian.
 - (c) reflexive, symmetric and transitive.
- (3) The gcd of the polynomials $f(x) = x^4 + 5x + 1$ and $g(x) = x^2 - 1$ in $\mathbb{F}_7[x]$ is:
 - (a) 1.
 - (b) $x^2 - 1$.
 - (c) $x - 1$.
 - (d) $x + 1$.

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- (4) Let $n \geq 1$ be an integer. Every homomorphism of groups $f : \mathbb{Z}/n\mathbb{Z} \rightarrow \mathbb{Z}$ is the zero homomorphism.
- (a) No.
 - (b) Yes.
 - (c) Depends on n .
- (5) There exists a transitive action of S_4 on the set $\{1, 2, 3, 4, 5\}$. (Recall that a transitive action of a group G on a set S is an action such that the following holds: for every $s_1, s_2 \in S$ there exists $g \in G$ such that $g * s_1 = s_2$.)
- (a) No.
 - (b) Yes.
- (6) Let B be a subset of two sets, A_1 and A_2 . If $|A_1 \setminus B| = |A_2 \setminus B|$ then the statement $|A_1| = |A_2|$ is:
- (a) true if A_1, A_2 are finite, but need not be true in general.
 - (b) may be false even if A_1, A_2 are finite.
 - (c) always true.
- (7) Let A, B, C be sets. Then $(A \cup B) \setminus (B \setminus C)$ is equal to:
- (a) $(A \cap C) \cup B$.
 - (b) $(A \cup B \cup C) \setminus (A \cap B \cap C)$.
 - (c) $(A \cup B) \setminus (B \cap C)$.
 - (d) $(A \setminus B) \cup (B \cap C)$.
- (8) Consider the following list of principal ideals $(2), (3), (5), (6)$ in the ring $\mathbb{Z}/14\mathbb{Z}$. There are
- (a) Three distinct ideals in this list.
 - (b) Two distinct ideals in this list.
 - (c) Only one ideal in this list.
 - (d) Four distinct ideals in this list.
- (9) $5^{91} \cdot (3^3 + 2)^{53}$ is congruent to:
- (a) $10 \pmod{19}$.
 - (b) $1 \pmod{19}$.
 - (c) $16 \pmod{19}$.

End of Part I

Part II. 64 points. 16 points per question. Supply complete solutions, clearly citing results you are using. The proofs, including all work, should be written neatly and carefully in the exam notebook. Points will be deducted for messy solutions.

- (1) Let \mathbb{F} be the field with 7 elements.
- (a) Write all the squares and cubes in \mathbb{F} .
 - (b) Construct a field \mathbb{L} with 7^2 elements. Show that both the polynomials $x^2 - 3$ and $x^2 + x - 1$ can be solved in \mathbb{L} and write down their solutions.
 - (c) Construct a field \mathbb{K} with 7^3 elements.
 - (d) (Bonus question) Show that the polynomial $x^2 - 3$ cannot be solved in \mathbb{K} .

- (2) Let m and n be positive integers such that $(n, m) = 1$.

- (a) Prove the Chinese Remainder Theorem:

$$\mathbb{Z}/mn\mathbb{Z} \cong \mathbb{Z}/m\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}.$$

- (b) Find all the solutions to the polynomial equation

$$x^2 + x + 11 \equiv 0 \pmod{221}$$

(The final answer should be given as congruence classes modulo 221. Note that $221 = 13 \times 17$.)

- (3) Prove Lagrange's theorem: Let G be a finite group and H a subgroup of G then the number of elements of H divides that of G :

$$\# H \mid \# G.$$

- (4) Find the number of necklace designs where the necklaces have 10 stones, 2 of which of type A , 4 of type B and 4 of type C . (The relevant set here has $\binom{10}{2} * \binom{8}{4} = 3150$ elements.) As always, we mean that we allow rotational and reflection symmetries.