1. For each of the following subsets, decide whether it is or is not a subspace of the given vector space (justify your answers using the three part subspace criterion):
   (a) The subset $\{z \in \mathcal{C} \big| |z| \leq 1\}$ of the complex vector space $\mathcal{C}$.
   (b) The subset of the real vector space of real valued functions of one variable consisting of differentiable functions.
   (c) The subset of the complex vector space of polynomials with complex coefficients consisting of those polynomials all of whose roots in $\mathcal{C}$ are distinct.
   (d) The subset of the real vector space of polynomials with real coefficients consisting of those polynomials of degree at most 10.
   (e) The intersection of any collection of subspaces of a vector space V.

2. Consider the polynomial $f(x) = x^3 - x^2 - 5x - 3 = (x-3)(x+1)^2$ and define $U_f = \{f(x)g(x) | g(x) \in \mathcal{F}[x]\}$ whereby $\mathcal{F}[x]$ consists of all polynomials in the indeterminate x with coefficients from the field $\mathcal{F}$.
   (a) Find polynomials $a(x),\ b(x)$ such that $a(x)(x-3) + b(x)(x+1)^2 = 1$
   (b) Show that the quotient space V/U_f is the internal direct sum of Image T_{x-3} and Image $T_{(x+1)^2}$ with the notation used in class. Why is $a(T)$ invertible on Image $T_{(x-3)}$?
   (c) Find a basis for the factor space such that the matrix of the induced linear transformation has the form
$$\begin{pmatrix} 3 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{pmatrix}$$

3. (a) Consider the set V of all sequences of elements of the field $\mathcal{F}$ as vectors with countably many components, e.g. $\mathbf{x} = (x_1, x_2, \ldots, x_k, \ldots)$ with all $x_j \in \mathcal{F}$ is a typical such vector.
   Show that the sequences satisfying the recurrence relation $x_{n+2} = ax_{n+1} + bx_n$ form a two dimensional subspace of V.
   (b) With the two dimensional subspace of the previous part of this question in mind, solve the following difference equation explicitly by reducing it to a problem about the eigenvalues of a two by two matrix. (That is, find an explicit, nonrecursive formula for x_n.)
$$x_1 = 5, \quad x_2 = 3, \quad x_{n+2} = 3x_{n+1} + 4x_n \text{ for } n \geq 1$$

4. (a) State and prove Cramer's rule for solving a system on n equations in n unknowns based on the three axioms given in class for the definition of the determinant function.
   (b) Given two three by three matrices A and B with entries from a field $\mathcal{F}$, define
$$C = \begin{pmatrix} A & \mathbf{0} \\ -I & B \end{pmatrix}$$
   whereby $\mathbf{0}$ is a three by three block of zeros and I is the three by three identity matrix.
      i. Using only column and row operations, show that $\det\ C = -\det \begin{pmatrix} -I & B \\ \mathbf{0} & AB \end{pmatrix}$.
      ii. Using column operations and/or the row cofactor expansion, conclude $\det\ A \det\ B = \det\ C = \det\ (AB)$

5. (a) Let $V = \mathcal{C}^4$ as an inner product space over the complex numbers $\mathcal{C}$ with the usual inner product and define $W = \text{Span}\{(1, i, -1, i)^t, (-i, 1, i, -1)^t\}$
      i. Find an orthonormal basis for $W^\perp$, the orthogonal complement of W.
      ii. Extend this basis to an orthonormal basis of V.
   (b) Let U be an inner product space over the complex numbers $\mathcal{C}$ and H a Hermitian operator $H : U \longrightarrow U$.
   Prove that eigenvectors of H for distinct eigenvalues are orthogonal.