MATH 223, LINEAR ALGEBRA

FALL 2009

FINAL EXAMINATION

Tuesday, December 8, 2009 9:00-12:00

Examiner: Professor Jim Loveys
Associate Examiner: Doctor Christophe Weibel

Instructions:

1. No notes, books or calculators permitted.
2. Do not write anything on the separate sheet summarizing the questions.
3. This exam has 8 questions. All questions carry the same weight.
4. Do all your work on the sheets provided. Do not separate sheets that have been stapled together. (YOU WILL LOSE MARKS IF YOU DO.)
5. The questions have been divided into two parts, purely to facilitate marking. **PART 1** of this exam consist of questions 1 to 4. **PART 2** of this exam consists of questions 5 to 8. Make sure you have a "white" and a "blue"set of questions. Make sure that your name, student number, and section number are on both parts. (If your instructor is Christophe Weibel, you are in section 1; if your instructor is Jim Loveys, you are in section 2.)

NAME:

STUDENT NUMBER:

SECTION NUMBER:

. DO NOT WRITE ANTHING BELOW HERE ON THIS PAGE.

QUESTION	1	2	3	4	TOTAL/40

This examination comprises this cover page, 8 pages of questions, and 8 extra (blank) pages. The question pages and blank pages are in two parts. There is also a sheet summarizing the problems.

1. Let $W_1 = Span\left\{\begin{pmatrix}1\\1\\1\\0\end{pmatrix}, \begin{pmatrix}1\\2\\1\\1\end{pmatrix}\right\}$ and $W_2 = Span\left\{\begin{pmatrix}1\\2\\3\\-4\end{pmatrix}, \begin{pmatrix}4\\6\\2\\7\end{pmatrix}\right\}$ be subspaces of $\mathcal{R}^4$. Find a basis for each of $W_1 + W_2$ and $W_1 \cap W_2$.

This page is for the continuation of problem 1; it may also be used for rough work.

2. Let $V = M_n(\mathcal{R})$ be the real vector space of $n \times n$ matrices with real entries and let A be a fixed $n \times n$ matrix with real entries. For $X \in V$, we define $TX = A^T XA$.

   (a) Verify that T is a linear operator on V.

   For the rest of the question $n = 2$ and $A = \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}$.

   (b) Let $B = \left(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right)$ be the standard ordered basis for V. Find $[T]_B$.

   (c) Find a basis for each of $ker(T)$ and $im(T)$.

This page is for the continuation of problem 2; it may also be used for rough work.

3. Let V be the vector space of functions defined and continuous on $[-1,1]$. For $f,g \in V$, we define $\langle f,g\rangle = \int_{-1}^{1} x^4 f(x)g(x)dx$.

   (a) Show that this defines an inner product on V.

   (b) Show that, for any $f \in V$,

   $$\left(\int_{-1}^{1} x^6 f(x)dx\right)^2 \leq \frac{2}{9}\int_{-1}^{1} x^4 f(x)^2 dx.$$

   Identify those functions f for which we have equality.

This page is for the continuation of problem 3; it may also be used for rough work.

4. Let V be a vector space and T a linear operator on V. Given that W is a subspace of V, recall that we say that W is *T-invariant* in case $T\vec{w} \in W$ for every $\vec{w} \in W$.

   Suppose that $p(x)$ is a polynomial and that W is T-invariant. Show that W is also $p(T)$-invariant. [Hint: First do this in case $p(x) = x^2$ and then when $p(x) = x^k$ for any k.]

This page is for the continuation of problem 4; it may also be used for rough work.