STUDENTS NAME **(underline Family name)**:
STUDENT NUMBER:
SECTION NUMBER:

McGILL UNIVERSITY

FACULTY OF SCIENCE

FINAL EXAMINATION

MATH 223

LINEAR ALGEBRA

Examiner: Professor O. Kharlampovich
Associate Examiner: Professor P. Russell

Date: Friday December 15, 2006
Time: 2:00PM - 5:00PM

INSTRUCTIONS

1. This is a closed book exam. No notes or books are permitted.
2. Simple Pocket Calculators that have no scientific functions are permitted only.
3. Do not write anything on the separate pink sheet summarizing the questions.
4. This exam has eight questions. All questions carry the same weight.
5. Do all your work on the sheets provided. Do not tear or separate sheets that have been stapled together.
6. The questions have been divided into two parts, purely to facilitate marking. Make sure you have a "white" and "blue" set of questions. Make sure that your name , student number and section number are on both parts. (If your instructor is Olga Kharlampovich , you are in section 1; if your instructor is Peter Russell, you are in section 2.)

This examination comprises of a cover page , 8 pages of questions , and 16 extra (blank) pages. The question pages and blank pages are in two parts. There is pink sheet summarizing the problems of this final exam that you may keep.

MATH 223, Linear Algebra, Fall 2006, Final Examination

PART ONE:

1. In this problem we work with vectors over the field $\mathbf{C}$ of complex numbers.

   Let $A = \begin{pmatrix} 1 & 0 & i \\ 0 & -1 & 1 \\ i & -1 & a \end{pmatrix}$,

   where $a \in \mathbf{C}$.

   (i) Find the row reduced echelon form A' of A.

   (ii) Show that A is singular precisely when $a = 0$.

   (iii) With $a = 0$, find bases for $Ker(A)$ and $Im(A)$. What is the dimension of the row space of A?

This page is for the continuation of problem 1; it may also be used for rough work.

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2. Let V_1 be the subspace of $\mathbf{R}^4$ spanned by

$$w_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, w_2 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, w_3 = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}.$$

Let $V_2 = V_1^{\perp}$, the orthogonal complement of V_1.

(i) Find an orthonormal basis for V_2.

(ii) Let L be the orthogonal projection of V to V_2. Find the matrix of L with respect to the standard basis of $\mathbf{R}^4$.

(iii) Give a geometric interpretation for the linear map $I - L$, where $I : V \to V$ is the identity map.

This page is for the continuation of problem 2; it may also be used for rough work.

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3. Let $V = P_2$, the vector space of real polynomials of degree at most two. Note that

$$B = \{1, t, t^2\}$$

is a basis of V.

(i) Show that

$$B' = \{1, t-1, (t-1)^2\}$$

is also a basis of V and find the change of basis matrix S from B to B', i.e the matrix S so that for all $p \in V$, $[p]_{B'} = S[p]_B$.

(ii) Let $T : V \to V$ be the map defined by $T(f)(t) = f(1)t$. Show that T is linear. Find the matrix that represents T in the basis B'.

This page is for the continuation of problem 3; it may also be used for rough work.

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4. Let V be the vector space of continuous real valued functions on the interval $[0,1]$. For $f, g \in V$ define

$$\langle f, g\rangle = \int_0^1 t f(t) g(t) dt.$$

(i) Show that $\langle -, -\rangle$ defines an inner product on V.

(ii) Let $U \subset V$ be the subspace spanned by the functions f_1, f_2 defined by $f_1(t) = 1, \quad f_2(t) = t$ for $t \in [0,1]$.
Compute the matrix $(\langle f_i, f_j\rangle)_{1\leq i,j\leq 2}$.

(iii) Find an orthonormal basis of U.

This page is for the continuation of problem 4; it may also be used for rough work.

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FINAL EXAMINATION - PART TWO

MATH 223

LINEAR ALGEBRA

This booklet consists of 4 pages of Questions (questions numbers 5 to 8) and 8 pages of blank sheets

5. For each of the following quadratic forms $Q(x_1, x_2, x_3)$, find $a, b, c \in \mathbf{R}$ so that $Q(x_1, x_2, x_3) = ay_1^2 + by_2^2 + cy_3^2$, where $\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$ is the coordinate vector of $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ with respect to a suitable orthonormal basis of $\mathbf{R}^3$.
(You do not have to compute this basis!)
Use this information to describe the shape of the graph of $Q(x_1, x_2, x_3) = 1$.

   (a) $Q(x_1, x_2, x_3) = 2x_1^2 + 2x_2^2 - 2x_1x_2 + 2x_3^2$;
   (b) $Q(x_1, x_2, x_3) = -x_1^2 + 10x_1x_3 + x_2^2 - x_3^2$.

This page is for the continuation of problem 5; it may also be used for rough work.

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6. Let $A = \begin{pmatrix} d & e & f \\ a & b & c \\ g & h & j \end{pmatrix}$ have determinant 2+2i, and $B = \begin{pmatrix} k & \ell & m \\ d & e & f \\ g & h & j \end{pmatrix}$ have determinant 3-i. Find the determinant of $\begin{pmatrix} a+3ik & d-g & 3g \\ b+3i\ell & e-h & 3h \\ c+3im & f-j & 3j \end{pmatrix}$.

This page is for the continuation of problem 6; it may also be used for rough work.

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7. Let $A = \begin{pmatrix} 1 & 2 & 1 \\ -4 & -9 & -4 \\ 7 & 16 & 7 \end{pmatrix}$. Does there exist an invertible matrix P such that $P^{-1}AP$ is diagonal? If it exists, find it.

This page is for the continuation of problem 7; it may also be used for rough work.

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8. Prove that

   a) all the eigenvalues of a Hermitian matrix are real (a complex matrix A is Hermitian if $A = A^* = \bar{A}^T$).

   b) the eigenvectors corresponding to different eigenvalues are orthogonal for the Hermitian inner product.

This page is for the continuation of problem 8; it may also be used for rough work.

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