McGILL UNIVERSITY

FACULTY OF SCIENCE

FINAL EXAMINATION

MATH 223-001 AND 002

LINEAR ALGEBRA

Examiner: Professor Kharlampovich — Date: Thurs. December 15, 2005

Associate Examiner: Professor Loveys — Time: 9:00 am - 12:00 pm

INSTRUCTIONS

(a) Answer questions in the exam booklets provided.

(b) All questions carry equal weight.

(c) This is a closed book exam. No computers, notes or text books are permitted.

(d) Simple pocket calculators that have no scientific functions are permitted only.

(e) Use of a regular and or translation dictionary is not permitted.

(f) This exam comprises of the cover page, 2 pages of 8 questions.

(g) EXAM IS PRINTED DOUBLE-SIDED

MATH 223 Final Examination Thursday December 15, 2005

Solve all problems. Faculty calculators are allowed. Books, notes are not allowed.

1. Find a basis for each of the row space, the column space, and the null space of the following matrix with entries in $\mathcal{C}$. What is its rank?

$$\begin{pmatrix} 1 & 1+i & 0 & 3 & 0 \\ 2i & -2+2i & 1 & 2+5i & -i \\ 1+i & 2i & -2i & 1-i & -2 \\ 3 & 3+3i & 1-i & 10-3i & -1-i \end{pmatrix}.$$

2. Let $W = Span\{\begin{pmatrix} 1 \\ 0 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 1 \\ 2 \end{pmatrix}\}$ be a subspace of $\mathcal{R}^4$. Find an orthonormal basis for each of W and $W^\perp$. Also find the projections $proj_W\vec{v}$ and $perp_W\vec{v}$ of $\vec{v}$ onto W and onto $W^\perp$, where $\vec{v} = \begin{pmatrix} 0 \\ 4 \\ 2 \\ 2 \end{pmatrix}$.

3. Let $P_3(t)$ be the vector space of polynomials over the reals with degree at most three. Let $T : P_3(t) \longrightarrow P_3(t)$ be defined by

$$Tf(t) = t^2 f''(t) - 3tf'(t) + 3f(t)$$

   (a) Find the matrix $[T]_{\mathcal{B}}$ of T with respect to the standard ordered basis $\mathcal{B} = \{1, t, t^2, t^3\}$ of $P_3(t)$. Do the same (i.e., find $[T]_{\mathcal{C}}$) for the nonstandard ordered basis $\mathcal{C} = (1, 1+t, 1+t+t^2, 1+t+t^2+t^3)$.

   (b) Find a basis for the kernel of T and for the image (range) of T.

4. Suppose that A and B are similar matrices, and λ is an eigenvalue of A. Show that λ is also an eigenvalue of B, and that the dimensions of the corresponding eigenspaces are the same for A and B.

5. Let $A = \begin{pmatrix} 8 & 18 & 8 \\ -4 & -9 & -4 \\ 1 & 2 & 1 \end{pmatrix}$. Find (explicitly) A^{25}. Find P such that $P^{-1}AP$ is diagonal.

6. Let V be the vector space of all continuous real-valued functions on the interval $[-\frac{\pi}{2}, \frac{\pi}{2}]$. For $f, g \in V$, define $< f, g >= \int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} \cos x f(x)g(x)dx$. Verify that this gives an inner product on V

7. Let $A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & j \end{pmatrix}$ have determinant $2+i$, and $B = \begin{pmatrix} k & \ell & m \\ d & e & f \\ g & h & j \end{pmatrix}$ have determinant $4-i$. Find the determinant of $\begin{pmatrix} a+2ik & d & 3g \\ b+2i\ell & e & 3h \\ c+2im & f & 3j \end{pmatrix}$.

8. For each of the following quadratic forms $Q(x_1, x_2, x_3)$, find an orthogonal substitution that diagonalizes Q. Identify the shape of the graphs of $Q(x_1, x_2) = 1$, $Q(x_1, x_2, x_3) = 1$.

   (a) $Q(x_1, x_2) = x_1^2 - 4x_1x_2 + x_2^2$;

   (b) $Q(x_1, x_2, x_3) = 3x_1^2 + 4x_1x_2 + 2x_2^2 - 4x_1x_3 + 4x_3^2$.