MATH 223, LINEAR ALGEBRA

FALL 2004

FINAL EXAMINATION

Thursday, December 16, 2004 14:00-17:00

Examiner: Professor Peter Russell
Associate Examiner: Professor James Loveys

Instructions:

1. No notes, books or calculators permitted.
2. Do not write anything on the separate sheet summarizing the questions.
3. This exam has eight questions. All questions carry the same weight.
4. Do all your work on the sheets provided. Do not separate sheets that have been stapled together.
5. The questions have been divided into two parts, purely to facilitate marking. Make sure you have a "white" and a "blue" set of questions. Make sure that your name, student number, and section number are on both parts. (If your instructor is Jim Loveys, you are in section 1; if your instructor is Peter Russell, you are in section 2.)

NAME:

STUDENT NUMBER:

SECTION NUMBER:

This examination comprises this cover page, 8 pages of questions, and 16 extra (blank) pages. The question pages and blank pages are in two parts. There is also a sheet summarizing the problems.

PART ONE:

1. Let $A = \begin{pmatrix} 1 & 1 & 5 & -2 & 1 \\ 2 & 2 & 10 & -3 & 3 \\ 4 & 4 & 20 & -9 & 3 \end{pmatrix}$.

   (a) Find a basis for each of the row space, the column space, and the null space of A.

   (b) Find an invertible matrix Q such that QA is in reduced row-echelon form.

This page is for the continuation of problem 1; it may also be used for rough work.

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2. For each of the following matrices A, find the characteristic polynomial χ_A and the minimal polynomial min_A. Find the eigenvalues, and a basis for each eigenspace. Decide in each case whether the matrix is diagonalizable over the reals.

   (a) $A = \begin{pmatrix} 0 & 2 & 1 \\ 2 & 3 & 2 \\ 1 & 2 & 0 \end{pmatrix}$.

   (b) $A = \begin{pmatrix} 3 & 1 & 0 \\ -1 & 3 & 0 \\ 0 & 0 & 2 \end{pmatrix}$.

   (c) $A = \begin{pmatrix} 6 & 0 & 12 \\ 0 & 4 & 0 \\ -3 & 0 & -6 \end{pmatrix}$.

This page is for the continuation of problem 2; it may also be used for rough work.

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3. Let V be the vector space $P_5(t)$ of polynomials over the reals of degree ≤ 5. Define $T : V \longrightarrow V$ by

$$T(p(t)) = t^2p''(t) - 2tp'(t) + 2p(t).$$

   (a) Show that T is linear.

   (b) Show that each of 1, t, t^2, t^3, t^4 and t^5 is an eigenvector of T. In each case, give the corresponding eigenvalue.

   (c) Find a basis for the kernel $ker(T)$ and the dimension $dim(Im(T))$ of the image of T.

This page is for the continuation of problem 3; it may also be used for rough work.

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4. Suppose that V is a finite-dimensional vector space, and $T : V \longrightarrow V$ is a linear operator on V such that $T^2 = T$, but T is not the zero operator or the identity operator.

   (a) Give an example of such a T. (You may let $V = \mathcal{R}^2$.)

   (b) Show that the minimal polynomial of T is $t^2 - t$. Show that T is diagonalizable.

   (c) Letting W_0 be the eigenspace corresponding to 0, and W_1 the eigenspace corresponding to 1, show that $V = W_0 \oplus W_1$; show also that $ker(T) = W_0$ and $Im(T) = W_1$.

   (d) In case $dim(V) = 3$, list all possible diagonal matrices which are $[T]_B$ for some ordered basis B of V.

This page is for the continuation of problem 4; it may also be used for rough work.

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PART TWO.

NAME:

STUDENT NUMBER:

SECTION NUMBER:

5. (a) Find

$$det \begin{pmatrix} a & -1 & 0 & 0 \\ -1 & a & -1 & 0 \\ 0 & -1 & a & -1 \\ 0 & 0 & -1 & a \end{pmatrix}.$$

(b) For which (complex) values of a is this matrix singular?

This page is for the continuation of problem 5; it may also be used for rough work.

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6. In this problem we use the standard Hermitian inner product on $\mathcal{C}^3$. We let $U \subseteq \mathcal{C}^3$ be the subspace spanned by $\{\vec{u}_1, \vec{u}_2\}$, where $\vec{u}_1 = \begin{pmatrix} 1 \\ i \\ -i \end{pmatrix}$ and $\vec{u}_2 = \begin{pmatrix} i \\ 0 \\ 2 \end{pmatrix}$.

   (a) Find an orthonormal basis for U.

   (b) If $\vec{v} = \begin{pmatrix} 0 \\ i \\ 0 \end{pmatrix}$, find the vector $\vec{u}$ in U such that the norm $||\vec{v} - \vec{u}||$ is as small as possible. What is this norm?

This page is for the continuation of problem 6; it may also be used for rough work.

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7. Let $V = \mathcal{R}^n$ be given the standard inner product — i.e., the dot product.

   (a) State the Cauchy-Schwarz inequality for this inner product space.

   (b) If $n \geq 2$ and $x_1, \ldots, x_n \in \mathcal{R}$, show that

$$\Big(\sum_{1<j<k\leq n} x_j x_k \Big)^2 \leq \frac{n-1}{2} \sum_{\ell=1}^{n} x_\ell^2 .$$

   [Hint: Apply your statement in part (a) with one vector having all its entries 1.]

This page is for the continuation of problem 7; it may also be used for rough work.

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8. (a) Let $A = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}$. Find an orthogonal matrix P and a diagonal matrix D such that $P^T A P = D$.

(b) Let

$$E = \{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in \mathcal{R}^3 : x_1^2 + x_2^2 + x_3^2 - x_1 x_2 - x_2 x_3 = 1 \}.$$

Find an orthonormal basis $B = \{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ of $\mathcal{R}^3$ and real numbers λ_1,λ_2 and λ_3 such that

$$E = \{ y_1 \vec{u}_1 + y_2 \vec{u}_2 + y_3 \vec{u}_3 : \lambda_1 y_1^2 + \lambda_2 y_2^2 + \lambda_3 y_3^2 = 1 \}.$$

This page is for the continuation of problem 8; it may also be used for rough work.

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MATH 223, Linear Algebra, Fall 2004, Final Examination Questions

For your convenience, on the two sides of this sheet, we list all the problems on this exam. You may keep this sheet, but do not write anything on it during the exam. All the problems are of equal weight.

1. Let $A = \begin{pmatrix} 1 & 1 & 5 & -2 & 1 \\ 2 & 2 & 10 & -3 & 3 \\ 4 & 4 & 20 & -9 & 3 \end{pmatrix}$.
   (a) Find a basis for each of the row space, the column space, and the null space of A.
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2. For each of the following matrices A, find the characteristic polynomial χ_A and the minimal polynomial min_A. Find the eigenvalues, and a basis for each eigenspace. Decide in each case whether the matrix is diagonalizable over the reals. $A = \begin{pmatrix} 0 & 2 & 1 \\ 2 & 3 & 2 \\ 1 & 2 & 0 \end{pmatrix}$; $A = \begin{pmatrix} 3 & 1 & 0 \\ -1 & 3 & 0 \\ 0 & 0 & 2 \end{pmatrix}$; $A = \begin{pmatrix} 6 & 0 & 12 \\ 0 & 4 & 0 \\ -3 & 0 & -6 \end{pmatrix}$.

3. Let V be the vector space $P_5(t)$ of polynomials over the reals of degree ≤ 5. Define $T : V \longrightarrow V$ by

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4. Suppose that V is a finite-dimensional vector space, and $T : V \longrightarrow V$ is a linear operator on V such that $T^2 = T$, but T is not the zero operator or the identity operator.
   (a) Give an example of such a T. (You may let $V = \mathcal{R}^2$.)
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(d) In case $dim(V) = 3$, list all possible diagonal matrices which are $[T]_B$ for some ordered basis B of V.

5. (a) Find

$$det \begin{pmatrix} a & -1 & 0 & 0 \\ -1 & a & -1 & 0 \\ 0 & -1 & a & -1 \\ 0 & 0 & -1 & a \end{pmatrix}.$$

(b) For which (complex) values of a is this matrix singular?

6. In this problem we use the standard Hermitian inner product on $\mathcal{C}^3$. We let $U \subseteq \mathcal{C}^3$ be the subspace spanned by $\{\vec{u}_1, \vec{u}_2\}$, where $\vec{u}_1 = \begin{pmatrix} 1 \\ i \\ -i \end{pmatrix}$ and $\vec{u}_2 = \begin{pmatrix} i \\ 0 \\ 2 \end{pmatrix}$.

(a) Find an orthonormal basis for U.

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7. Let $V = \mathcal{R}^n$ be given the standard inner product — i.e., the dot product.

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(b) Let $E = \{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in \mathcal{R}^3 : x_1^2 + x_2^2 + x_3^2 - x_1 x_2 - x_2 x_3 = 1\}$. Find an orthonormal basis $B = \{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ of $\mathcal{R}^3$ and real numbers λ_1,λ_2 and λ_3 such that $E = \{y_1\vec{u}_1 + y_2\vec{u}_2 + y_3\vec{u}_3 : \lambda_1 y_1^2 + \lambda_2 y_2^2 + \lambda_3 y_3^2 = 1\}$.