McGill University

Faculty of Science

MATH 204

PRINCIPLES OF STATISTICS II

Final Examination

Date: 17th April 2008 Time: 2pm-5pm

Examiner: Dr. David A. Stephens
Associate Examiner: Dr. Russell Steele

Please write your answers in the answer booklets provided.

This paper contains six questions. Each question carries 25 marks. Credit will be given for all questions attempted. The total mark available is 150 but rescaling of the final mark may occur.

Candidates may take one double-sided sheet of Letter-sized (216 $\times$ 279 mm) or A4-sized (210 $\times$ 297mm) paper with handwritten notes into the examination room.

Calculators may be used. Relevant statistical tables are provided.

Dictionaries and Translation dictionaries are permitted.

1. In an experimental study of the efficacy of treatments for depression, patients attending a clinic were allocated at random to one of a number different drug treatments and other therapies. The response measurement was the decrease in Hamilton Rating Depression Score (HRDS) over a three month course of treatment. The data recorded are tabulated below; entries in the rows of the table are the decrease in HRDS for each treatment. The final group was treated using an inactive treatment (or placebo).

Group					
Therapy 1	5.20	3.65	7.03	5.63	4.57
Therapy 2	7.05	7.91	1.46	4.11	4.18
Drug A	4.71	3.75	2.85	4.89	6.09
Drug B	6.82	6.77	8.31	6.57	6.55
Placebo	5.13	6.07	3.55	5.69	3.97

(a) What kind of design is being used in this study ? Identify the *factor(s)* being investigated, and for each factor state the number of levels that that factor has. Is the study *balanced* ? Is it *complete* ? Justify your answers.

6 MARKS

(b) Using the data, an ANOVA analysis is to be carried out. The ANOVA table below contains some missing entries marked by the notation $\star$. Write out the ANOVA table in full, filling in the missing values **using the information already given in the table**.

SOURCE	DF	SS	MS	F
GROUP	$\star$	19.613	$\star$	$\star$
ERROR	$\star$	$\star$	2.295	
TOTAL	$\star$	$\star$		

10 MARKS

(c) What is the conclusion of the ANOVA analysis ? State clearly the null and alternative hypothesis, the test statistic, the null distribution, and the conclusion.

A table of the Fisher-F distribution is provided on page 15. Entries in the table are the 0.05 tail quantile of the Fisher-F(ν_1, ν_2) distribution, for different values of ν_1 and ν_2.

5 MARKS

(d) The following table was also produced during an SPSS analysis of the data.

Test of Homogeneity of Variances

Y

Levene Statistic	df1	df2	Sig.
2.800	4	20	.054

Explain the relevance of this table in relation to the ANOVA. What steps would you carry out to confirm that the conclusion of the ANOVA is valid ?

4 MARKS

2. (a) Explain how to construct a complete balanced randomized block design to investigate the effect of treatment TMT (a factor taking k levels) in the presence of a blocking factor B (a factor taking b levels).

4 MARKS

(b) Using standard model notation, where, for example,

$$\text{TMT} + \text{B}$$

indicates the main effects only model with factors TMT and B, list the four models that can be fitted if the design does not have replication. Give the **total** number of parameters that each of the four models contains.

4 MARKS

(c) The efficacy of a blood pressure reduction treatment is to be investigated in an experimental study. Four dose levels (labelled 1,2,3,4) were investigated in three groups of patients (labelled 1,2,3, corresponding to 40-49 years, 50-59 years, over 60 years respectively); it was strongly believed before the study was carried out that the efficacy of the treatments would be different depending on the age of the patient. The recorded response was the reduction in blood pressure over four weeks of treatment.

The data that were collected in this complete and balanced design were analyzed in SPSS; the output is included on page 8. The variable **agegp** denotes the different age groups, whereas **tmt** denotes dose level.

(i) Explain the results presented in the ANOVA table.

6 MARKS

(ii) In terms of mean responses for the different subgroups of patients, give an interpretation of the interaction term, **agegp*tmt**. Use a sketch if necessary.

6 MARKS

(d) Using the Parameter Estimates table given on page 8, report an estimate of the difference in expected response between patients in Age-Group 40-49 at dose level 1 and patients in Age Group 60+ at dose level 4.

Remember that SPSS uses a **contrast parameterization**, with the highest level of a factor determining the baseline category.

5 MARKS

3. An experiment was carried out to investigate whether two drugs, levorphanol and epinephrine, can be used to reduce stress in adult and juvenile mice. Injections of one, other or both treatments were given to four adult and four juvenile animals, and the cortical sterone level (which reflects the stress-level) was measured. The data recorded are given in the table below. Note that measurements on three animals could not be made and are omitted from the table.

	Adult		Juvenile	
	Epinephrine		Epinephrine	
Levorphanol	No	Yes	No	Yes
No	1.11,0.84,1.47,4.45	1.67,1.53,2.84	3.24,1.89,2.17,2.25	3.02,1.74,3.62
Yes	0.17,1.41,-0.18,-0.17	1.54,2.60,2.22,2.63	0.39,-0.43,0.67,1.13	2.23,3.42,2.42

Output from the SPSS analysis of the response data is given on pages 9 and 10.

(a) What is the name of this kind of design ?

3 MARKS

(b) Using standard model notation, list the five models that were fitted in the SPSS analysis. Use L,E and A for the main effects of factor levorphanol, epinephrine and adult respectively.

6 MARKS

(c) Using the SPSS output, find the most appropriate model in ANOVA-F terms using **backward selection**. Recall that the ANOVA-F test for comparing nested models is based on the statistic

$$F = \frac{(\text{SSE}_R - \text{SSE}_C)/(k-g)}{\text{SSE}_C/(n-k-1)}$$

where

- SSE_R is the error sum of squares for the **Reduced Model**, specified using $g+1$ parameters including the intercept.
- SSE_C is the error sum of squares for the **Complete Model**, specified using $k+1$ parameters including the intercept.

If the reduced model is an adequate simplification of the complete model, then

$$F \sim \text{Fisher-F}(k-g, n-k-1)$$

Comment on the overall model fit of your selected model.

10 MARKS

(d) Explain why the p-values given within the ANOVA tables cannot be used formally to decide whether terms should be omitted from or included in the model.

2 MARKS

(e) Explain how to reanalyze the data for the **full factorial** model using a *permutation* (or *randomization*) procedure that would overcome the issue raised in part (d). Recall that in such a procedure, the permuted treatment sample sizes should be identical to those in the original experiment.

4 MARKS

4. The gas consumption of 56 houses was measured over a month, and the average outdoor temperature at the houses was also measured. Two types of houses were studied; 26 did not have cavity wall insulation, and 30 did have insulation.

Output from an SPSS analysis is recorded on pages 11 and 12. In the analysis, gas consumption is the response, **Insul** is a factor predictor for insulation (coded as 0 for no insulation, 1 for insulation) and **Temp** is a continuous covariate.

(a) Explain, using sketches, the five different models that can be fitted to the response that involve main effects and interactions involving **Insul** and **Temp**.

5 MARKS

(b) Using the results presented in the SPSS output, explain the most appropriate model to explain the observed response. Comment on the global fit of the model.

6 MARKS

(c) Comment on the residual plot given on page 12 obtained from the first model fitted. How does this residual plot assist in the analysis ? What other plots would you need ?

6 MARKS

(d) Basing your answer on the fit of the model

$$\textbf{Insul} + \textbf{Temp} + \textbf{Insul.Temp} \tag{1}$$

compute an estimate, s^2, of the error variance, σ^2 in the regression model.

4 MARKS

(e) Basing your answer on the fit of the model in equation (1) predict the expected gas consumption for an uninsulated house in a month where the average temperature is 5 Celsius. By how much does the prediction change if the temperature is 6 Celsius ?

4 MARKS

5. (a) Explain how to carry out a Chi-squared test for **independence** of assignment of two factors, A and B, each of which takes two levels, using the statistic

$$X^2 = \sum_{i=1}^{2}\sum_{j=1}^{2} \frac{(n_{ij} - \widehat{n}_{ij})^2}{\widehat{n}_{ij}}$$

for a sample size of n. Explain what the terms in X^2 are, how to complete the test using tables of the Chi-squared distribution, and any assumptions that are required to hold in order for the test result to be valid.

8 MARKS

(b) Carry out a Chi-squared test of independence for the following 2×2 table that records the counts in a case-control study carried out to examine a hypothesized link between excema and hay fever in a sample of 11 year old children.

	Hay Fever		
Excema	Yes	No	Total
Yes	141	420	561
No	928	13525	14453
Total	1069	13945	15014

The table on page 16 contains the 0.05 and 0.01 tail quantiles of the Chisquared(ν) distribution, for $\nu = 1$ to 20.

6 MARKS

(c) In a first phase of a health study in a city, a random sample of size 2000 is to be obtained. The city is comprised (broadly) of five different ethnic subpopulations that make up 40 %, 30 %, 10 %, 10 % and 10% of the city population respectively.

A commercial company is employed to obtain the random sample, with the instruction that the sample should reflect the ethnic composition of the city. The sample they return is summarized in the following table.

	Ethnic Subpopulation				
	1	2	3	4	5
Expected Proportion (%)	40	30	10	10	10
Number in Sample	822	638	210	157	173

Using a Chi-squared test for this one-way layout, comment on whether the company have fulfilled their remit to produce a sample that reflects the ethnic composition of the city.

6 MARKS

(d) Using the usual guidelines under which the Chi-squared test is regarded to be valid, what is the smallest total sample size for which the test in part (c) could be considered ? Justify your answer.

5 MARKS

6. (a) Outline the differences between *paired* and *unpaired* (also known as *dependent* and *independent*) two sample tests in terms of the method of data collection.

4 MARKS

(b) In the table below, the measured central venous oxygen, SvO_2, in a group of 10 intensive care unit (ICU) patients, and 12 healthy controls is recorded. The study design involved measuring SvO_2 for the ICU patients at the time of admission (Time 0) and then six hours later (Time 1); for the healthy controls, two measurements were taken six hours apart. For all patients, the difference between the two measurements was computed.

ICU Patients			Healthy Controls		
Time 0	Time 1	Diff	Time 0	Time 1	Diff
59.1	56.7	-2.4	67.3	63.0	-4.3
58.2	60.7	2.5	64.8	64.3	-0.5
56.0	59.5	3.5	67.4	65.5	-1.9
65.3	59.8	-5.5	65.4	68.9	3.5
56.1	61.9	5.8	67.6	66.0	-1.6
60.6	67.7	7.1	63.7	65.1	1.4
37.8	50.0	12.2	64.5	68.2	3.7
39.7	52.9	13.2	68.2	63.3	-4.9
57.7	71.4	13.7	69.8	65.9	-3.9
33.6	51.3	17.7	64.3	66.6	2.3
			68.9	68.7	-0.2
			70.5	65.8	-4.7

After reading the SPSS output on pages 13 and 14 **carefully**, give answers to the following questions, and give supporting details of the statistical test results. In each case, indicate whether the test is **paired** or **unpaired**. Note that in most cases, the **asymptotic** test result only is quoted. **Note also that some of the SPSS output corresponds to inappropriate tests**.

(i) Do patients in the ICU exhibit a significant **increase** in SvO_2 level over the six hour period ?

5 MARKS

(ii) Is the amount by which SvO_2 changes over the six hour period **different for ICU patients and compared to healthy controls** ?

5 MARKS

(iii) Does SvO_2 level **change** significantly in healthy control subjects over a six hour period ?

5 MARKS

(c) Suppose that, in a new experimental study, three ICUs were to be compared to discover whether they were equally adept at increasing the SvO_2 measurement of patients in their care. Identify (by name) the type of design such a study would require, and identify one parametric test, and one non-parametric test, that could be used to assess the hypothesis of interest.

6 MARKS

Output for Question 2

Tests of Between-Subjects Effects

Dependent Variable:Y

Source	Type III Sum of Squares	df	Mean Square	F	Sig.
Corrected Model	197.112[a]	11	17.919	4.357	.000
Intercept	1346.978	1	1346.978	327.517	.000
agegp	63.231	2	31.616	7.687	.001
tmt	57.541	3	19.180	4.664	.006
agegp * tmt	76.340	6	12.723	3.094	.012
Error	197.409	48	4.113		
Total	1741.500	60			
Corrected Total	394.521	59			

a. R Squared = .500 (Adjusted R Squared = .385)

Parameter Estimates

Dependent Variable:Y

Parameter	B	Std. Error	t	Sig.	95% Confidence Interval	
					Lower Bound	Upper Bound
Intercept	8.780	.907	9.680	.000	6.956	10.603
[agegp=1]	-4.602	1.283	-3.588	.001	-7.181	-2.024
[agegp=2]	-3.732	1.283	-2.910	.005	-6.311	-1.153
[agegp=3]	0[a]	.	.	.	.	.
[tmt=1]	-3.240	1.283	-2.526	.015	-5.819	-.662
[tmt=2]	-4.315	1.283	-3.364	.002	-6.894	-1.736
[tmt=3]	-2.829	1.283	-2.206	.032	-5.408	-.251
[tmt=4]	0[a]	.	.	.	.	.
[agegp=1] * [tmt=1]	5.785	1.814	3.189	.003	2.138	9.432
[agegp=1] * [tmt=2]	2.133	1.814	1.176	.245	-1.514	5.780
[agegp=1] * [tmt=3]	1.343	1.814	.740	.463	-2.304	4.990
[agegp=1] * [tmt=4]	0[a]	.	.	.	.	.
[agegp=2] * [tmt=1]	1.199	1.814	.661	.512	-2.448	4.847
[agegp=2] * [tmt=2]	2.686	1.814	1.481	.145	-.961	6.333
[agegp=2] * [tmt=3]	2.848	1.814	1.570	.123	-.799	6.495
[agegp=2] * [tmt=4]	0[a]	.	.	.	.	.
[agegp=3] * [tmt=1]	0[a]	.	.	.	.	.
[agegp=3] * [tmt=2]	0[a]	.	.	.	.	.
[agegp=3] * [tmt=3]	0[a]	.	.	.	.	.
[agegp=3] * [tmt=4]	0[a]	.	.	.	.	.

a. This parameter is set to zero because it is redundant.

Output for Question 3: Part I

Tests of Between-Subjects Effects

Dependent Variable:Stress Reduction

Source	Type III Sum of Squares	df	Mean Square	F	Sig.
Corrected Model	24.099[a]	7	3.443	4.280	.004
Intercept	97.952	1	97.952	121.773	.000
lev	5.380	1	5.380	6.688	.017
epi	9.578	1	9.578	11.907	.002
adult	1.398	1	1.398	1.739	.202
epi * adult	.196	1	.196	.243	.627
lev * adult	.177	1	.177	.220	.644
lev * epi	6.224	1	6.224	7.738	.011
lev * epi * adult	.001	1	.001	.001	.973
Error	16.892	21	.804		
Total	133.830	29			
Corrected Total	40.991	28			

a. R Squared = .588 (Adjusted R Squared = .451)

Tests of Between-Subjects Effects

Dependent Variable:Stress Reduction

Source	Type III Sum of Squares	df	Mean Square	F	Sig.
Corrected Model	23.738[a]	4	5.935	8.256	.000
Intercept	98.312	1	98.312	136.761	.000
lev	5.410	1	5.410	7.526	.011
epi	9.605	1	9.605	13.362	.001
adult	1.275	1	1.275	1.773	.195
lev * epi	6.240	1	6.240	8.681	.007
Error	17.253	24	.719		
Total	133.830	29			
Corrected Total	40.991	28			

a. R Squared = .579 (Adjusted R Squared = .509)

Output for Question 3: Part II

Tests of Between-Subjects Effects

Dependent Variable:Stress Reduction

Source	Type III Sum of Squares	df	Mean Square	F	Sig.
Corrected Model	22.464[a]	3	7.488	10.104	.000
Intercept	97.642	1	97.642	131.753	.000
lev	5.605	1	5.605	7.564	.011
epi	9.370	1	9.370	12.644	.002
lev * epi	6.049	1	6.049	8.162	.008
Error	18.527	25	.741		
Total	133.830	29			
Corrected Total	40.991	28			

a. R Squared = .548 (Adjusted R Squared = .494)

Tests of Between-Subjects Effects

Dependent Variable:Stress Reduction

Source	Type III Sum of Squares	df	Mean Square	F	Sig.
Corrected Model	16.415[a]	2	8.207	8.683	.001
Intercept	99.902	1	99.902	105.688	.000
lev	6.992	1	6.992	7.397	.011
epi	10.040	1	10.040	10.622	.003
Error	24.577	26	.945		
Total	133.830	29			
Corrected Total	40.991	28			

a. R Squared = .400 (Adjusted R Squared = .354)

Tests of Between-Subjects Effects

Dependent Variable:Stress Reduction

Source	Type III Sum of Squares	df	Mean Square	F	Sig.
Corrected Model	6.374[a]	1	6.374	4.972	.034
Intercept	94.412	1	94.412	73.639	.000
lev	6.374	1	6.374	4.972	.034
Error	34.617	27	1.282		
Total	133.830	29			
Corrected Total	40.991	28			

a. R Squared = .156 (Adjusted R Squared = .124)

Output for Question 4: Part I

Between-Subjects Factors

		Value Label	N
Insulation	0	Before	26
	1	After	30

Tests of Between-Subjects Effects

Dependent Variable:Gas Consumption (1000 cu. ft.)

Source	Type III Sum of Squares	df	Mean Square	F	Sig.
Corrected Model	69.589[a]	3	23.196	222.333	.000
Intercept	431.193	1	431.193	4132.905	.000
Insul	14.594	1	14.594	139.882	.000
Temp	45.577	1	45.577	436.850	.000
Insul * Temp	1.345	1	1.345	12.893	.001
Error	5.425	52	.104		
Total	1003.300	56			
Corrected Total	75.014	55			

a. R Squared = .928 (Adjusted R Squared = .924)

Parameter Estimates

Dependent Variable:Gas Consumption (1000 cu. ft.)

Parameter	B	Std. Error	t	Sig.	95% Confidence Interval Lower Bound	95% Confidence Interval Upper Bound
Intercept	4.724	.118	40.000	.000	4.487	4.961
[Insul=0]	2.130	.180	11.827	.000	1.769	2.491
[Insul=1]	0[a]	.	.	.	.	.
Temp	-.278	.023	-12.124	.000	-.324	-.232
[Insul=0] * Temp	-.115	.032	-3.591	.001	-.180	-.051
[Insul=1] * Temp	0[a]	.	.	.	.	.

a. This parameter is set to zero because it is redundant.

Output for Question 4: Part II

Tests of Between-Subjects Effects

Dependent Variable:Gas Consumption (1000 cu. ft.)

Source	Type III Sum of Squares	df	Mean Square	F	Sig.
Corrected Model	68.244[a]	2	34.122	267.114	.000
Intercept	429.865	1	429.865	3365.077	.000
Insul	33.224	1	33.224	260.088	.000
Temp	45.896	1	45.896	359.286	.000
Error	6.770	53	.128		
Total	1003.300	56			
Corrected Total	75.014	55			

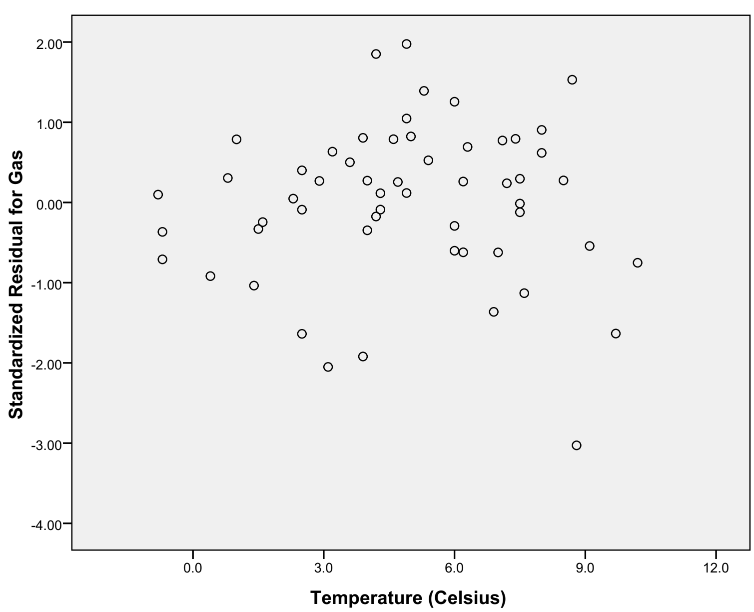
a. R Squared = .910 (Adjusted R Squared = .906)

Parameter Estimates

Dependent Variable:Gas Consumption (1000 cu. ft.)

Parameter	B	Std. Error	t	Sig.	95% Confidence Interval	
					Lower Bound	Upper Bound
Intercept	4.986	.103	48.558	.000	4.780	5.192
[Insul=0]	1.565	.097	16.127	.000	1.371	1.760
[Insul=1]	0[a]	.	.	.	.	.
Temp	-.337	.018	-18.955	.000	-.372	-.301

a. This parameter is set to zero because it is redundant.



Output for Question 6: Part I

Analysis 1: Mann-Whitney Test

Ranks

	Time	N	Mean Rank	Sum of Ranks
Sv02	Time 0	22	21.61	475.50
	Time 1	22	23.39	514.50
	Total	44		

Test Statistics[a]

	Sv02 at admission
Mann-Whitney U	222.500
Wilcoxon W	475.500
Z	-.458
Asymp. Sig. (2-tailed)	.647

a. Grouping Variable: Time

Analysis 2: Wilcoxon Signed Ranks Test

Ranks

		N	Mean Rank	Sum of Ranks
ICUTIME1 - ICUTIME0	Negative Ranks	2[a]	2.50	5.00
	Positive Ranks	8[b]	6.25	50.00
	Ties	0[c]		
	Total	10		

a. ICUTIME1 < ICUTIME0

b. ICUTIME1 > ICUTIME0

c. ICUTIME1 = ICUTIME0

Test Statistics[b]

	ICUTIME1 - ICUTIME0
Z	-2.293[a]
Asymp. Sig. (2-tailed)	.022

a. Based on negative ranks.

b. Wilcoxon Signed Ranks Test

Analysis 3 : Wilcoxon Signed Ranks Test

Ranks

		N	Mean Rank	Sum of Ranks
HEATIME1 - HEATIME0	Negative Ranks	8[a]	6.75	54.00
	Positive Ranks	4[b]	6.00	24.00
	Ties	0[c]		
	Total	12		

a. HEATIME1 < HEATIME0

b. HEATIME1 > HEATIME0

c. HEATIME1 = HEATIME0

Test Statistics[b]

	HEATIME1 - HEATIME0
Z	-1.177[a]
Asymp. Sig. (2-tailed)	.239

a. Based on positive ranks.

b. Wilcoxon Signed Ranks Test

Output for Question 6: Part II

Analysis 4 : Wilcoxon Signed Ranks Test

Ranks

		N	Mean Rank	Sum of Ranks
Sv02 after 6 hours - Sv02 at admission	Negative Ranks	10[a]	8.90	89.00
	Positive Ranks	12[b]	13.67	164.00
	Ties	0[c]		
	Total	22		

a. Sv02 after 6 hours < Sv02 at admission

b. Sv02 after 6 hours > Sv02 at admission

c. Sv02 after 6 hours = Sv02 at admission

Test Statistics[b]

	Sv02 after 6 hours - Sv02 at admission
Z	-1.218[a]
Asymp. Sig. (2-tailed)	.223

a. Based on negative ranks.

b. Wilcoxon Signed Ranks Test

Analysis 5 : Mann-Whitney Test

Ranks

	Patie...	N	Mean Rank	Sum of Ranks
Difference	ICU	10	15.15	151.50
	Healthy	12	8.46	101.50
	Total	22		

Test Statistics[b]

	Difference
Mann-Whitney U	23.500
Wilcoxon W	101.500
Z	-2.407
Asymp. Sig. (2-tailed)	.016
Exact Sig. [2*(1-tailed Sig.)]	.014[a]

a. Not corrected for ties.

b. Grouping Variable: Patient Group

Table of the Fisher-F distribution

Entries in table are the $\alpha = 0.05$ tail quantile of Fisher-F(ν_1, ν_2) distribution
ν_1 given in columns, ν_2 given in rows.

$\nu_2 \backslash \nu_1$	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25
1	161.45	199.50	215.71	224.58	230.16	233.99	236.77	238.88	240.54	241.88	242.98	243.91	244.69	245.36	245.95	246.46	246.92	247.32	247.69	248.01	248.31	248.58	248.83	249.05	249.26
2	18.51	19.00	19.16	19.25	19.30	19.33	19.35	19.37	19.38	19.40	19.40	19.41	19.42	19.42	19.43	19.43	19.44	19.44	19.44	19.45	19.45	19.45	19.45	19.45	19.46
3	10.13	9.55	9.28	9.12	9.01	8.94	8.89	8.85	8.81	8.79	8.76	8.74	8.73	8.71	8.70	8.69	8.68	8.67	8.67	8.66	8.65	8.65	8.64	8.64	8.63
4	7.71	6.94	6.59	6.39	6.26	6.16	6.09	6.04	6.00	5.96	5.94	5.91	5.89	5.87	5.86	5.84	5.83	5.82	5.81	5.80	5.79	5.79	5.78	5.77	5.77
5	6.61	5.79	5.41	5.19	5.05	4.95	4.88	4.82	4.77	4.74	4.70	4.68	4.66	4.64	4.62	4.60	4.59	4.58	4.57	4.56	4.55	4.54	4.53	4.53	4.52
6	5.99	5.14	4.76	4.53	4.39	4.28	4.21	4.15	4.10	4.06	4.03	4.00	3.98	3.96	3.94	3.92	3.91	3.90	3.88	3.87	3.86	3.86	3.85	3.84	3.83
7	5.59	4.74	4.35	4.12	3.97	3.87	3.79	3.73	3.68	3.64	3.60	3.57	3.55	3.53	3.51	3.49	3.48	3.47	3.46	3.44	3.43	3.43	3.42	3.41	3.40
8	5.32	4.46	4.07	3.84	3.69	3.58	3.50	3.44	3.39	3.35	3.31	3.28	3.26	3.24	3.22	3.20	3.19	3.17	3.16	3.15	3.14	3.13	3.12	3.12	3.11
9	5.12	4.26	3.86	3.63	3.48	3.37	3.29	3.23	3.18	3.14	3.10	3.07	3.05	3.03	3.01	2.99	2.97	2.96	2.95	2.94	2.93	2.92	2.91	2.90	2.89
10	4.96	4.10	3.71	3.48	3.33	3.22	3.14	3.07	3.02	2.98	2.94	2.91	2.89	2.86	2.85	2.83	2.81	2.80	2.79	2.77	2.76	2.75	2.75	2.74	2.73
11	4.84	3.98	3.59	3.36	3.20	3.09	3.01	2.95	2.90	2.85	2.82	2.79	2.76	2.74	2.72	2.70	2.69	2.67	2.66	2.65	2.64	2.63	2.62	2.61	2.60
12	4.75	3.89	3.49	3.26	3.11	3.00	2.91	2.85	2.80	2.75	2.72	2.69	2.66	2.64	2.62	2.60	2.58	2.57	2.56	2.54	2.53	2.52	2.51	2.51	2.50
13	4.67	3.81	3.41	3.18	3.03	2.92	2.83	2.77	2.71	2.67	2.63	2.60	2.58	2.55	2.53	2.51	2.50	2.48	2.47	2.46	2.45	2.44	2.43	2.42	2.41
14	4.60	3.74	3.34	3.11	2.96	2.85	2.76	2.70	2.65	2.60	2.57	2.53	2.51	2.48	2.46	2.44	2.43	2.41	2.40	2.39	2.38	2.37	2.36	2.35	2.34
15	4.54	3.68	3.29	3.06	2.90	2.79	2.71	2.64	2.59	2.54	2.51	2.48	2.45	2.42	2.40	2.38	2.37	2.35	2.34	2.33	2.32	2.31	2.30	2.29	2.28
16	4.49	3.63	3.24	3.01	2.85	2.74	2.66	2.59	2.54	2.49	2.46	2.42	2.40	2.37	2.35	2.33	2.32	2.30	2.29	2.28	2.26	2.25	2.24	2.24	2.23
17	4.45	3.59	3.20	2.96	2.81	2.70	2.61	2.55	2.49	2.45	2.41	2.38	2.35	2.33	2.31	2.29	2.27	2.26	2.24	2.23	2.22	2.21	2.20	2.19	2.18
18	4.41	3.55	3.16	2.93	2.77	2.66	2.58	2.51	2.46	2.41	2.37	2.34	2.31	2.29	2.27	2.25	2.23	2.22	2.20	2.19	2.18	2.17	2.16	2.15	2.14
19	4.38	3.52	3.13	2.90	2.74	2.63	2.54	2.48	2.42	2.38	2.34	2.31	2.28	2.26	2.23	2.21	2.20	2.18	2.17	2.16	2.14	2.13	2.12	2.11	2.11
20	4.35	3.49	3.10	2.87	2.71	2.60	2.51	2.45	2.39	2.35	2.31	2.28	2.25	2.22	2.20	2.18	2.17	2.15	2.14	2.12	2.11	2.10	2.09	2.08	2.07
21	4.32	3.47	3.07	2.84	2.68	2.57	2.49	2.42	2.37	2.32	2.28	2.25	2.22	2.20	2.18	2.16	2.14	2.12	2.11	2.10	2.08	2.07	2.06	2.05	2.05
22	4.30	3.44	3.05	2.82	2.66	2.55	2.46	2.40	2.34	2.30	2.26	2.23	2.20	2.17	2.15	2.13	2.11	2.10	2.08	2.07	2.06	2.05	2.04	2.03	2.02
23	4.28	3.42	3.03	2.80	2.64	2.53	2.44	2.37	2.32	2.27	2.24	2.20	2.18	2.15	2.13	2.11	2.09	2.08	2.06	2.05	2.04	2.02	2.01	2.01	2.00
24	4.26	3.40	3.01	2.78	2.62	2.51	2.42	2.36	2.30	2.25	2.22	2.18	2.15	2.13	2.11	2.09	2.07	2.05	2.04	2.03	2.01	2.00	1.99	1.98	1.97
25	4.24	3.39	2.99	2.76	2.60	2.49	2.40	2.34	2.28	2.24	2.20	2.16	2.14	2.11	2.09	2.07	2.05	2.04	2.02	2.01	2.00	1.98	1.97	1.96	1.96
26	4.23	3.37	2.98	2.74	2.59	2.47	2.39	2.32	2.27	2.22	2.18	2.15	2.12	2.09	2.07	2.05	2.03	2.02	2.00	1.99	1.98	1.97	1.96	1.95	1.94
27	4.21	3.35	2.96	2.73	2.57	2.46	2.37	2.31	2.25	2.20	2.17	2.13	2.10	2.08	2.06	2.04	2.02	2.00	1.99	1.97	1.96	1.95	1.94	1.93	1.92
28	4.20	3.34	2.95	2.71	2.56	2.45	2.36	2.29	2.24	2.19	2.15	2.12	2.09	2.06	2.04	2.02	2.00	1.99	1.97	1.96	1.95	1.93	1.92	1.91	1.91
29	4.18	3.33	2.93	2.70	2.55	2.43	2.35	2.28	2.22	2.18	2.14	2.10	2.08	2.05	2.03	2.01	1.99	1.97	1.96	1.94	1.93	1.92	1.91	1.90	1.89
30	4.17	3.32	2.92	2.69	2.53	2.42	2.33	2.27	2.21	2.16	2.13	2.09	2.06	2.04	2.01	1.99	1.98	1.96	1.95	1.93	1.92	1.91	1.90	1.89	1.88
31	4.16	3.30	2.91	2.68	2.52	2.41	2.32	2.25	2.20	2.15	2.11	2.08	2.05	2.03	2.00	1.98	1.96	1.95	1.93	1.92	1.91	1.90	1.88	1.88	1.87
32	4.15	3.29	2.90	2.67	2.51	2.40	2.31	2.24	2.19	2.14	2.10	2.07	2.04	2.01	1.99	1.97	1.95	1.94	1.92	1.91	1.90	1.88	1.87	1.86	1.85

Table of the Chisquared(ν) distribution

Entries in table are the $\alpha = 0.05$ and $\alpha = 0.01$ tail quantiles

ν	1	2	3	4	5	6	7	8	9	10
$\alpha = 0.05$	3.841	5.991	7.815	9.488	11.070	12.592	14.067	15.507	16.919	18.307
$\alpha = 0.01$	6.635	9.210	11.345	13.277	15.086	16.812	18.475	20.090	21.666	23.209

ν	11	12	13	14	15	16	17	18	19	20
$\alpha = 0.05$	19.675	21.026	22.362	23.685	24.996	26.296	27.587	28.869	30.144	31.410
$\alpha = 0.01$	24.725	26.217	27.688	29.141	30.578	32.000	33.409	34.805	36.191	37.566