NAME:

STUDENT NUMBER:

McGILL UNIVERSITY

FACULTY OF SCIENCE

FINAL EXAMINATION

MATH 133

Vectors Matrices and Geometry

Examiner: Professor J. Labute
Associate Examiner: Professor I. Klemes

Date: Monday, December 17, 2007
Time: 9:00-12:00 NOON

INSTRUCTIONS

Attempt all questions.
All questions are of equal value.
Answer all questions on the pages provided.
Show and justify all your work.
Calculators, books and notes are not permitted.
A translation dictionary is permitted.
All matrices are real matrices.

This exam comprises the cover, 10 pages with 10 questions and 6 additional blank pages.

1	2	3	4	5	6	7	8	9	10

1. The plane Π has vector equation

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -4 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + t \begin{bmatrix} -3 \\ 0 \\ 4 \end{bmatrix}.$$

(a) Find an equation $ax_1 + bx_2 + cx_3 = d$ for the plane Π.

(b) Find the point Q in the plane $2x + 3y + z = 10$ which is closest to the point $P(7, 7, 3)$.

2. (a) Find the equation of the line passing through the points $A(1,2,3)$ and $B(2,1,5)$.

(b) Find the distance between the line in part (a) and the line $x = 2 - 2t, y = 4 + 2t, z = 7 - 4t$.

3. Let A be the matrix

$$A = \begin{bmatrix} 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 4 & 4 & 4 \\ 1 & 2 & -3 & -8 & 0 \\ 1 & 2 & -1 & -6 & 2 \end{bmatrix}.$$

(a) Bring A to reduced row echelon form. Clearly indicate each of the elementary operations that you use.

(b) Find bases for the row space, column space and null space of A.

4. (a) Prove or disprove the following statement:

$$\text{Span}\{[1,2,-1,-2],[2,1,2,-1]\} = \text{Span}\{[-1,4,-7,-4],[8,7,4,-7]\}.$$

(b) If $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are linearly independent vectors in $\mathbb{R}^n$, for which values of k are the vectors $k\mathbf{u}+\mathbf{v}, \mathbf{v}+k\mathbf{w}, \mathbf{w}+k\mathbf{u}$ linearly independent?

5. (a) Let $R : \mathbb{R}^2 \to \mathbb{R}^2$ be the reflection in the line $2x + 5y = 0$. Find two linearly independent eigenvectors of R and give their corresponding eigenvalues. You may use either the standard matrix of R or geometric reasoning.

(b) Find the standard matrix A of the linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ determined by the conditions

$$T\left(\begin{bmatrix} 2 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} -3 \\ 6 \end{bmatrix}, \quad T\left(\begin{bmatrix} 5 \\ 3 \end{bmatrix}\right) = \begin{bmatrix} -2 \\ 1 \end{bmatrix}.$$

6. Let

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 3 \end{bmatrix}.$$

(a) Find the inverse of A and write A^{-1} as a product of elementary matrices.

(b) Write A as a product of elementary matrices.

7. (a) Let A be an invertible 3×3 matrix. Suppose it is known that

$$A = \begin{bmatrix} u & v & w \\ 3 & 3 & -2 \\ x & y & z \end{bmatrix} \quad \text{and that} \quad \text{adj}(A) = \begin{bmatrix} a & 3 & b \\ -1 & 1 & 2 \\ c & -2 & d \end{bmatrix}.$$

Find $\det(A)$. (Give an answer not involving any of the unknown variables.)

(b) If A is a matrix such that $A^2 - A + I = 0$ show that A is invertible with inverse $I - A$.

8. Let $A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$.

(a) Find the eigenvalues of A and a basis for each of its eigenspaces.

(b) Find an invertible matrix P such that $P^{-1}AP$ is a diagonal matrix.

9. (a) For which values of k is the matrix $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & k \end{bmatrix}$ diagonalizable?

(b) Let A and B be diagonalizable 2×2 matrices. If every eigenvector of A is an eigenvector of B show that $AB = BA$.

10. Let $q(\mathbf{X}) = 3x_1^2 + 2x_1x_2 + 3x_2^2$.

(a) Find an orthogonal change of coordinates $\mathbf{X} = P\mathbf{Y}$ such that $q(\mathbf{X}) = ay_1^2 + by_2^2$ for suitable scalars a, b.

(b) Find the maximum and minimum values of q on the circle $||\mathbf{X}|| = 1$.