1. Let $f : \mathbb{R}^3 \to \mathbb{R}^4$ be given by

$$F(x,y,z) = (x^2 - y^2,\ xy,\ xz,\ yz), \quad (x,y,z) \in \mathbb{R}^3.$$

Let $S^2 \subset \mathbb{R}^3$ be the unit sphere centered at the origin. Show that the map $\varphi = F\Big|_{S^2}$ induces a well-defined map $\tilde{\varphi} : \mathbb{P}_2(\mathbb{R}) \to \mathbb{R}^4$. Prove that $\tilde{\varphi}$ is an immersion. Is it an embedding? Justify all your answers carefully.

2. Use the Mayer-Vietoris sequence to compute the de Rham cohomology of $\mathbb{R}^2 \backslash \{(0,0),(0,1)\}$.

3. Compute the de Rham cohomology of the Lie group $SL(2,\mathbb{R})$.

4. Consider the Poincaré upper half-plane (M,g), where $M = \{(x,y) \in \mathbb{R}^2 | y > 0\}$ and

$$g\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial x}\right) = g\left(\frac{\partial}{\partial y}, \frac{\partial}{\partial y}\right) = \frac{1}{y^2}, \quad g\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right) = 0.$$

Putting $z = x + iy$, show that the transformation f given by $z \mapsto \dfrac{az+b}{cz+d}$, $a,b,c,d \in \mathbb{R}$, $ad - bc = 1$ is a global isometry of (M,g), that is a diffeomorphism of M satisfying

$$f^*g = g.$$

5. Consider again the Poincaré upper half-plane (M,g), as in problem 4. Consider the map $\gamma : (a,b) \to M$, $a > 0$, $t \mapsto (0,t)$. Show that the image of γ can be parametrized as a geodesic curve. Use the isometries determined in problem 4 to transform the image of γ into parts of semi-circles. Are these also images of geodesic curves? Please justify your answer.

(Remark: These semi-circles are called horocycles.)

McGILL UNIVERSITY

FACULTY OF SCIENCE

FINAL EXAMINATION

MATHEMATICS 189-577B

GEOMETRY AND TOPOLOGY II

Examiner: Professor N. Kamran	Date: Wednesday, April 19, 2000
Associate Examiner: Professor P. Russell	Time: 2:00 P.M. - 5:00 P.M.

INSTRUCTIONS

Calculators are not permitted.

This exam comprises the cover and one page of questions.