1. This problem does not require that you prove anything. For each of the following six partial phase portraits, determine which are correct and complete them. Determine which are impossible, modify them not by deleting any orbits shown but by changing stability types of existing orbits, or fixed points or adding new orbits or fixed points.

2. Determine the phase portrait of the system

$$\dot{x} = y$$
$$\dot{y} = x - 2x^3 + \lambda y(x^4 - x^2 + y^2)$$

for all values of the scalar parameter λ. What are the possible limit points?

3. Consider the following vector fields, with a parameter $\mu \in \mathbb{R}$.

   (a) $\dot{r} = -r(r-\mu)^2, \quad \dot{\theta} = 1, \qquad (r,\theta) \in \mathbf{R}^+ \times S^1.$

   (b) $\dot{r} = r(\mu - r^2)(2\mu - r^2)^2, \quad \dot{\theta} = 1.$

   (c) $\dot{r} = r(r+\mu)(r-\mu), \quad \dot{\theta} = 1.$

   (d) $\dot{r} = \mu r(r+\mu)^2, \quad \dot{\theta} = 1.$

   (e) $\dot{r} = -\mu^2 r(r+\mu)^2(r-\mu)^2, \quad \dot{\theta} = 1.$

   Match each of these vector fields to the appropriate phase portrait in Figure A and explain which hypotheses (if any) of the Hopf bifurcation theorem are violated. (Hint: This is <u>not</u> a computation intensive exercise.)

4. Study the dynamics near the origin for the following vector field. Draw the phase portraits. Compute the center manifold and describe the dynamics on the center manifold. Discuss the stability or instability of the origin.

   (a) $\dot{x} = -y - y^3$

   (b) $\dot{y} = 2x.$

5. Consider the following parametrized families of vector fields with parameter $\varepsilon \in \mathbf{R}^1$. For $\varepsilon = 0$, the origin is a fixed point of each vector field. Study the dynamics near the origin for ε small. Draw phase portraits. Compute the one-parameter family of center manifolds and describe the dynamics on the center manifolds. How do the dynamics depend on ε? Discuss the role played by a parameter by comparing these cases. In, for example (a) and (b), the parameter ε multiplies a linear and nonlinear term, respectively. Discuss the differences in these two cases in the most general setting possible.

   (a) $\dot{x} = -y - \varepsilon x - y^3, \quad \dot{y} = 2x$

   (b) $\dot{x} = -y - y^3, \quad \dot{y} = 2x + \varepsilon x^2.$

6. Redo problem 8 from the midterm, using the following additional hint:
   Consider the region bounded by $0 \leq x \leq 1,\ x = 1,,\ y = ax$ for a suitably chosen $a < 0$ and show that it is invariant. Construct a Liapounov function on this region.

McGILL UNIVERSITY

FACULTY OF SCIENCE

TAKE-HOME EXAMINATION

MATHEMATICS 189-574A

ORDINARY DIFFERENTIAL EQUATIONS

Examiner: Professor R. Rigelhof — Date: Friday, December 18, 1998

INSTRUCTIONS

This exam is due Friday, December 18, 1998 at 17:00 and may be handed in at BURN 1222 or at the departmental office on the tenth floor.

This exam comprises the cover, 2 pages of questions and one diagram.