1. An oil refinery has available three different processes to produce gasoline. Each process produces varying amounts of three grades of gasoline: regular, low-lead and premium. These amounts, in hundreds of gallons per hour of operation, are given in the following table, along with the cost in dollars of an hour's operation of each of the processes:

	Regular	Low-Lead	Premium	Cost
Process 1	3	4	2	160
Process 2	6	6	8	400
Process 3	6	3	4	300

The refinery must meet the weekly demands (in hundreds of gallons) of 36 gal of regular, 20 gal of low-lead, and 30 gal of premium.

(a) Formulate a mathematical programming model that should determine an operation of the refinery that satisfies the above demands and minimizes costs. Consider this model as a "primal" program. Do not solve this model!

(b) Give a dual formulation of the model from (a). Solve this dual using the simplex method. Denote an optimal solution of the dual by u^*. Determine the optimal weekly operation x^* of the refinery (i.e., an optimal solution of the primal program) from the final simplex table of the dual.

(c) Identify the most sensitive technological coefficient in the matrix (i.e., the process and the type of gasoline which cause biggest changes in the optimal value function).

2. Consider the linear Pareto problem with two objective functions:

$$\begin{array}{ll} \text{Max} & \{x_1 + 2x_2,\ 3x_1 + 4x_2\} \\ \text{s.t.} & \\ & x_1 + x_2 \leq 10 \\ & x_1 \leq 5 \\ & x_1 \geq 0,\ x_2 \geq 0. \end{array}$$

Using the Charnes-Cooper observation and the Karush-Kuhn-Tucker conditions check whether $x^* = (5,5)^T$ is a Pareto maximum.

3. In order to find a local minimum of a twice continuously differentiable function $f : \mathbb{R}^n \to \mathbb{R}$ one can use any of the three basic methods: Cauchy method of steepest descent, Newton's method, and a quasi-Newton method.

(a) Explain how each of these methods moves from x^k to a better point x^{k+1}.

(b) How are the gradients $\nabla f(x^k)$ and $\nabla f(x^{k+1})$, obtained in two successive iterations, related to each other in the methods of Cauchy and quasi-Newton? Prove your claims. (It is not required to give formulas for updatings of the Hessian.)

4. Consider the program

$$\begin{aligned} \text{Max} \quad & (7.6 - 0.04x_1)x_1 + (5 - 0.02x_2)x_2 \\ & 0.2x_1 + 0.2x_2 \leq 20 \\ & 0.8x_1 + 0.3x_2 \leq 60 \\ & x_1 \geq 0, \; x_2 \geq 0. \end{aligned}$$

   (a) A feasible point of this program is $x^0 = (0,0)^T$. Using an optimality condition show that this point is not optimal. Then find a better feasible point (i.e., one that gives a higher value to the objective function) using a method of feasible directions. (Perform only one full iteration in R^2, i.e., find x^1.) Find the corresponding step size accurately to at least 3 decimal places using a method of your choice.

   (b) An optimal solution of this program appears to be $x_1^* = 55$ and $x_2^* = 45$. Check this statement using an optimality condition.

5. Consider a twice continuously differentiable function $f : \mathbb{R}^n \to \mathbb{R}$. It is well known that a point x^* is an isolated minimum of f if $\nabla f(x^*) = 0$ and if $\nabla^2 f(x^*)$ is a positive definite matrix.

   (a) Prove this statement.

   (b) How does this result extend to the problems with equality constraints such as

$$\begin{aligned} \text{Min} \quad & f(x) \\ \text{s.t.} \quad & \\ & h^i(x) = 0, \; i \in P? \end{aligned}$$

   Illustrate your claim on the following problem (obtained in solving a Zermelo problem):

$$\begin{aligned} \text{Min} \quad & t \\ \text{s.t.} \quad & \\ & (3t\cos\theta - 2t + 10)^2 + (3t\sin\theta - 5)^2 = 1. \end{aligned}$$

   Here it is claimed that $\theta^* = 2.416$ and $t^* = 2.179$ is an isolated local minimum? Is this true?

6. (**For 189-487A version**) Consider a linear program on canonical form:

$$\begin{aligned} \text{Max} \quad & c^T x \\ \text{s.t.} \quad & \\ & Ax = b, \; x \geq 0 \end{aligned}$$

   where A is an $m \times n$ matrix of rank m. Assume that the feasible set is not empty. Denote by N an $n \times (n-p)$ matrix whose columns form a basis of the null-space of A. Joey, a promising student of mathematical programming, claims that the following statement is correct: "A feasible point x^* is an optimal solution of this program if, and only if, the system

$$N^T(u + c) = 0, \; u^T x^* = 0, \; u \geq 0$$

   is consistent (in u)." Joey claims that he can prove this result by applying the Karush-Kuhn-Tucker theorem to the above linear program. Is Joey right?

McGILL UNIVERSITY

FACULTY OF SCIENCE

FINAL EXAMINATION

MATHEMATICS 189-417A/487A

MATHEMATICAL PROGRAMMING

Examiner: Professor S. Zlobec — Date: Wednesday, December 16, 1998
Associate Examiner: Professor N. Sancho — Time: 2:00 P.M. - 5:00 P.M.

INSTRUCTIONS

Students taking 189-417A exam must attempt Problems 1,2,3,4, and 5.
Students taking 189-487A exam must attempt Problems 1,2,3,4, and 6.
Non-programmable calculators are allowed.

This exam comprises the cover and 2 pages of questions.