1. (a) Factor the polynomial $p(z) = z^3 + 1 + i$ as a product of linear factors, using polar form for the roots. Sketch a plot of the roots in the complex plane.

   (b) Find the Principal Value of $\log(-1 - i\sqrt{3})$.

   (c) Determine whether or not

$$\lim_{z \to 0} \frac{\overline{z}}{z}$$

   exists and prove your assertion.

2. (a) Define the term Isolated Singularity of a function $f(z)$, and the term $\text{Res}(f(z), a)$ where $z = a$ is an isolated singularity of $f(z)$. Then find the residues of

   (b) $\dfrac{z^{1/2}}{z^3 - 4z^2 + 4z}$ at all isolated singularities, using the principal value of $z^{1/2}$.

   (c) $\dfrac{z^2}{1 - e^z}$ at all isolated singularities.

3. Let the function f be defined by

$$f(t) = \frac{\sin 2t}{t} \quad , \; -\infty < t < \infty, \; t \neq 0,$$

   and set $f(0) = 2$. Evaluate its Fourier Transform $F(\omega)$ for $-\infty < \omega < \infty$ (except at jump discontinuities of F),using contour integration. Include the detailed analysis of each part of your contour.

4. Given the complex function

$$F(s) = \frac{e^{-3s}}{s^2 + s + 1},$$

   (a) Find its Inverse Laplace Transform $f(t)$, $-\infty < t < \infty$, using contour integration. Include diagrams of your contours for the cases $t > 3$ and $t < 3$.

   (b) Write down the integral for $F(s)$ in terms of $f(t)$ (the Laplace Transform). Determine the set of all complex numbers s for which this integral converges absolutely, and sketch the set in the $s-$plane.

5. For the complex function $f(z) = z^{-2}(z - 1)^{-1}(z + 3)^{-1}$,

   (a) State the three domains in which a Laurent Series in powers of z is available. Is the series a Taylor series in any of these three cases ?

   (b) Define Principal Part of a function f near an isolated singularity $z = a$. Determine the Principal Part of the given f near $z = 0$.

   (c) Compute the Laurent Series refered to in (a) for the domain containing the point $\sqrt{2} + i\sqrt{2}$.

6. (a) For $f(t) = 1 + 3^t,\ t \geq 0$, find the Z transform $F(z)$. Find all singularities of $F(z)$.

   (b) For $G(z) = z^{-3}(z-4)^{-1}$, find the inverse Z transform $g(nT),\ n \geq 0$.

   (c) Write down the series for $G(z)$ in terms of $g(nT)$ in part (b) (the Z transform). Determine the set of all complex numbers z for which the series converges absolutely and sketch this set in the $z-$plane.

7. Let $f(z) = z^3 + z^2 + 3z + 16$.

   (a) Prove that if $|z| \geq 10$ then $f(z) \neq 0$. Hint: find a lower bound on $|f(z)|$.

   (b) Let γ be the closed contour shown below, consisting of the line segment from $10i$ to $-10i$ plus the semicircle of radius 10 in the right half plane, traversed counterclockwise. Given that the image of γ under the mapping $z \to f(z)$ is as shown below, how many zeros of $f(z)$ are on γ ? inside γ ? Explain.

   (c) How many zeros of $f(z)$ are there in the whole right half plane ? in the left half plane ? on the y axis ? (Assume the results in (a) and (b) are correct).

FACULTY OF ENGINEERING

FINAL EXAMINATION

MATHEMATICS 189-381B

COMPLEX VARIABLES AND TRANSFORMS

Examiner: Professor I. Klemes
Associate Examiner: Professor D. Sussman

Date: Thursday, April 29, 1999
Time: 9:00 A.M. - 12:00 Noon

INSTRUCTIONS

NO CALCULATORS PERMITTED
Show all work and simplify answers.
Answer all 7 questions.
Keep this exam paper.

This exam comprises the cover and 2 pages of questions.