1. Determine all the plane curves $\vec{\alpha}(s)$, $s =$ arc length, such that the angle between $\vec{\alpha}(s)$ and $\vec{T}(s) = \vec{\alpha}'(s)$ is a constant θ, $0 < \theta < \pi$.

2. Let $\vec{\alpha}(s)$, $s =$ arc length, be a curve whose torsion τ is a non-zero constant, say $\tau = 1/a$. Show that $\vec{\alpha}$ can be expressed in the form

$$\vec{\alpha}(s) = a \int \vec{g}(s) \times \vec{g}'(s)\, ds$$

for some vector-valued function $\vec{g}(s)$ satisfying $|\vec{g}(s)| = 1$ and $(\vec{g} \times \vec{g}') \cdot \vec{g}'' \neq 0$.

3. Let $k_n(\theta)$ denote the normal curvature of a surface M in the direction of $\vec{u} = \cos\theta \vec{e}_1 + \sin\theta \vec{e}_2$, where $\vec{e}_1$ and $\vec{e}_2$ are principal tangent vectors in the tangent plane T_pM. Show that the mean curvature $H(p)$ of M at p is given by

$$H(p) = \frac{1}{2\pi} \int_0^{2\pi} k_n(\theta) d\theta.$$

4. Compute the Gaussian curvature of the surface $z = e^{-\frac{1}{2}(x^2+y^2)}$, $(x, y) \in \mathbb{R}^2$. Sketch this surface indicating the regions where $K < 0$, $K = 0$ and $K > 0$.

5. Determine all the surfaces of revolution

$$\vec{X}(u, v) = (g(u),\, h(u)\cos v,\, h(u)\sin v),$$

(where $h(u) > 0$ and the profile curve $(g(u), h(u), 0)$ has unit speed) of constant negative Gaussian curvature $K = -\dfrac{1}{c^2}$.

6. Let $\vec{X}(u, v)$ be a patch. A parallel surface to $\vec{X}$ at distance a is defined by the patch

$$\vec{Y}(u, v) = \vec{X}(u, v) + a\vec{N}(u, v)$$

where a is a constant.

7. (a) Show that

$$\vec{Y}_u \times \vec{Y}_v = (1 - 2Ha + Ka^2)(\vec{X}_u \times \vec{X}_v),$$

where H and K denote the mean and Gaussian curvatures of $\vec{X}$.

(b) Show that at regular points the Gaussian curvature of $\vec{Y}$ is given by

$$K_{\vec{Y}} = \frac{K}{1 - 2Ha + Ka^2}$$

and the mean curvature of $\vec{Y}$ is given by

$$H_{\vec{Y}} = \frac{H - Ka}{1 - 2Ha + Ka^2}.$$

(c) Suppose that $\vec{X}$ has non-zero Gaussian curvature K. Suppose furthermore that $\vec{X}$ has constant mean curvature equal to $c \neq 0$. Show that the parallel surface at distance $a = \dfrac{1}{2c}$ has constant Gaussian curvature equal to $4c^2$.

McGILL UNIVERSITY

FACULTY OF SCIENCE

FINAL EXAMINATION

MATHEMATICS 189-380B

Differential Geometry of Curves & Surfaces

Examiner: Professor N. Kamran — Date: Thursday, April 27, 2000
Associate Examiner: Professor H. Darmon — Time: 2:00 P.M. - 5:00 P.M.

This exam comprises the cover and two pages of questions.