Final Examination
Mathematics 189-377B
Number Theory

Justify all your assertions

Part I

1. (a) If $r_1, ..., r_{p-1}$ is a reduced residue system modulo a prime p, show that

$$\prod_{j=1}^{p-1} \equiv -1 \pmod{p}.$$

(b) If $r_1, ..., r_{p-1}$ and $s_1, ..., s_{p-1}$ are reduced residue systems modulo an odd prime p, show that $r_1 s_1, ..., r_{p-1} s_{p-1}$ is not a reduced residue system modulo p.

2. (a) State and prove the Chinese Remainder Theorem.

(b) Find all solutions of the system

$$4x \equiv 6 \pmod{22}$$
$$3x \equiv 7 \pmod{10}.$$

3. (a) Find all solutions of $x^2 \equiv 2 \pmod{343}$.

(b) Find all solutions of $x^2 \equiv 2 \pmod{119}$.

4. (a) Let p be an odd prime. Show that if g is a primitive root modulo p^2 then g is a primitive root modulo p^3.

(b) Find a primitive root modulo 343.

Part II

5. (a) State the Law of Quadratic Reciprocity and state and prove the Lemma of Gauss that is used to prove it.

(b) Find all primes p such that 10 is a square modulo p.

6. Using the fact that $4001x^2 + 6204xy + 2405y^2$ is a quadratic form with discriminant -4, find a representation of 4001 as a sum of two squares.

7. Find all integer solutions of the system

$$x + 2y + 4z = 3$$
$$2x + 7y - z = -6.$$

8. (a) Let f, g, h be arithmetic functions such that $h(n) = \sum_{d|n} f(d)g(n/d)$ for all n. Show that if f, g are multiplicative then so is h.

(b) Show that $1/\varphi(n) = \frac{1}{n} \sum_{d|n} \mu(d)^2/\varphi(d)$.