1. (6 marks) Consider a homogeneous solid body with density δ, specific heat c, conductivity K, all assumed constant. If the rate of heat generation is $Q(\vec{r},t)$ in ca./c.c./sec., say, show that the temperature $\psi(\vec{r},t)$ at a point $P(\vec{r})$ after time t satisfies the partial differential equation
$$\frac{1}{\alpha^2}\frac{\partial\psi}{\partial t} - \nabla^2\psi = \frac{Q}{K}$$
where $\alpha^2 = \dfrac{K}{c\delta}$.

2. (12 marks) Solve
$$\psi_t - \psi_{xx} = h(x,t);\;\; 0 < x < \pi,\;\; t > 0$$
   (a) $\psi_x(0,t) = \left[\dfrac{\partial\psi}{\partial x}(x,t)\right]_{x=0} = F(t)$.
   (b) $\psi_x(L,t) = \left[\dfrac{\partial\psi}{\partial x}(x,t)\right]_{x=L} = G(t)$.
   (c) $\psi(x,0) = f(x)$

   <u>and</u> interpret physically.

3. (a) (8 marks) Find the potential distribution inside a hemisphere if the spherical part is maintained at a potential $f(\cos\phi)$ and the flat part is insulated. <u>Hint</u>: Show that the insulation of the flat face, i.e.
$$\left[\frac{\partial\psi}{\partial z}\right]_{\phi=\pi/2} = 0 \text{ implies } \left[\frac{\partial\psi}{\partial\phi}\right]_{\phi=\pi/2} = 0.$$
   (b) (4 marks) Consider the special case in (a) of $\psi(a,\phi) = V_0(1 + 2\sin^2\phi)$, with V_0 a constant, giving your answer in <u>simplest</u> form. Note "a" is the radius of the hemisphere.

   You may assume that the general solution of Laplace's equation in spherical coordinates, i.e., $\nabla^2\psi(r,\varphi) = 0,\; 0 \le \varphi \le \pi,\; \psi$ finite at $\varphi = 0$ and $\varphi = \pi$, is given by $\psi(r,\varphi) = \displaystyle\sum_{n=0}^{\infty}\left[A_n r^n + \frac{B_n}{r^{(n+1)}}\right]P_n(\cos\varphi)$, where P_n are the Legendre polynomials of order n.

4. (11 marks) Find the steady-state temperature distribution in the region below:

5. (12 marks) Solve and interpret physically:

   (a) $\nabla^2\psi(r,z) = 0; \quad 0 < r < b,\ 0 < z < \pi.$

   (i) $\psi(r,0) = 0,$ (ii) $\psi(r,\pi) = f(r),$ (iii) $\psi(b,z) = g(z).$

   Hint: Divide the problem into two parts.

   (b) (9 marks) $\nabla^2\psi(r,z) = -F(r,z); \quad 0 < r < b,\ 0 < z < \pi.$

   (i) $\psi(r,0) = 0,$ (ii) $\psi(r,\pi) = f(r),$ (iii) $\psi(b,z) = g(z).$

6. (12 marks) A sphere of radius b has its surface maintained at a constant temperature of T°. There is a constant heat generation at the rate of Q (cal./c.c./sec., say) inside the sphere. If the initial temperature is $f(r)$ determine the temperature at any point inside the sphere after time t.

   Hint: The temperature $\psi = \psi(r,t)$ only.

7. (a) (8 marks) Solve the boundary value problem:

$$\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} = 0, \qquad 0 < x < \infty,\ 0 < y < \infty$$

   (i) $\psi(x,0) = 0,$ (ii) $\lim_{x\to\infty}\psi(x,y) = 0,$ (iii) $\left[\dfrac{\partial\psi}{\partial x}\right]_{x=0} \begin{array}{l} = 0,\ y > b \\ = -q/K,\ 0 < y < b. \end{array}$

   You may leave your answer as a single integral. Interpret physically.

   (b) (5 marks) Show that the magnitude of the heat current through the face $y = 0$, i.e., $K\left[\dfrac{\partial\psi}{\partial y}\right]_{y=0}$, where K is the conductivity, is given by $\dfrac{q}{\pi}\ln\left[1 + \dfrac{b^2}{x^2}\right]$.

8. (13 marks) Solve the following Dirichlet problem:

$$\frac{\partial\psi}{\partial t} = \frac{\partial^2\psi}{\partial x^2} + h(x,t);\ 0 < x < \infty,\ t > 0.$$

(i) $\psi(x,0) = f(x),$ (ii) $\psi(0,t) = q(t),$ (iii) $\lim_{x\to\infty}\psi(x,t) = 0,$ (iv) $\lim_{x\to\infty}\psi_x(x,t) = 0.$

Good Luck!

FACULTY OF SCIENCE

FINAL EXAMINATION

MATHEMATICS 189-375A
DIFFERENTIAL EQUATIONS

PHYSICS 198-355A
MATHEMATICAL PHYSICS

Examiner: Professor C. Roth
Associate Examiner: Professor D. Sussman

Date: Friday, December 19, 1997
Time: 2:00 P.M. - 6:00 P.M.

INSTRUCTIONS

Calculators are neither required nor permitted.

This exam comprises the cover, 2 pages of questions and 1 page of useful information.