1. Let $E \subset \mathbb{R}$ be a bounded, Lebesgue measurable set. Given $\varepsilon > 0$, show that there is an open set G and a compact set C with

   (a) $C \subseteq E \subseteq G$

   (b) $m(G \backslash C) < \varepsilon$.

   Hence show that for any Lebesgue measurable subset E of $\mathbb{R}$

$$m(E) = \sup\{m(C) : C \text{ compact}, \ C \subseteq E\} = \inf\{m(G) : G \text{ open}, \ G \supseteq E\}.$$

2. Let (X, S, μ) be a measure space, $f \geq 0$, measurable on X. If $\nu_f(E) = \int_E f \, d\mu$, we know that $\mu(E) = 0$ implies $\nu_f(E) = 0$. If f is integrable on X prove the stronger statement that given $\varepsilon > 0$ there exists $\delta > 0$ such that $\mu(E) < \delta$ implies $\nu_f(E) < \varepsilon$.

3. Find $\lim_{n \to \infty} \int_0^\infty n \sin\left(\frac{x}{n}\right) [x(1+x^2)]^{-1} dx$.

   [Justify carefully all interchanges of limits.]

4. State the theorems of Fubini and of Fubini-Tonnelli on the integration of functions measurable with respect to the product of two σ-finite measure spaces $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \nu)$. Show that

$$\int_0^1 \left\{ \int_0^1 \frac{x^2 - y^2}{(x^2 + y^2)^2} dy \right\} dx \neq \int_0^1 \left\{ \int_0^1 \frac{x^2 - y^2}{(x^2 + y^2)^2} dx \right\} dy$$

   although both sides exist. $\left[\underline{\text{Hint}}: \frac{\partial}{\partial y} \left(\frac{y}{x^2 + y^2} \right) = \frac{x^2 - y^2}{(x^2 + y^2)^2}. \right]$

   What does Fubini's theorem then tell you?

5. (a) State Hölder's inequality. Let (X, S, μ) be a finite measure space with $\mu(X) = 1$, and $f \in L^p$ for some p, $1 < p < \infty$. Use Hölder's inequality to show that $f \in L^1$ and $\|f\|_1 \leq \|f\|_p$.

   (b) If $f_n,\ f \in L^p(X, S, \mu),\ g_n,\ g \in L^q(X, S, \mu),\ 1 < p < \infty,\ q = \dfrac{p}{p-1}$, and $f_n \to f$ in L^p, $g_n \to g$ in L^q show that

$$f_n g_n \to fg \text{ in } L^1 \text{ and } \int_X f_n g_n d\mu \to \int_X fg d\mu.$$

6. Let (X, S, μ) be a measure space, (ϕ_n), $n \in \mathbb{N}$ an orthonormal sequence in $L^2(X, S, \mu)$, and $(c_n) \in \ell^2$.

   State the Riesz-Fischer theorem for (ϕ_n) and (c_n). Now for each $n \in \mathbb{N}$, put $s_n = \sum_{j=1}^{n} c_j \phi_j$ and $\gamma_n = \sum_{j=n}^{\infty} |c_j|^2$. If (n_i), $i \in \mathbb{N}$ is a strictly increasing subsequence of $\mathbb{N}$ such that $\sum_{i=1}^{\infty} \gamma_{n_i}$ converges, show that (s_{n_i}) converges a.e. μ as $i \to \infty$.

   [Hint: Use the Riesz-Fischer Theorem to identify the limit, and then, together with the sum form of the Monotone Convergence Theorem to show that $(s_{n_i - 1})$ converges a.e. μ as $i \to \infty$. A similar, but simpler argument shows that $c_n \phi_n \to 0$ a.e. μ as $n \to \infty$.]

7. Prove or disprove the following, i.e. if the statement is true, give a proof, if it is false give a counterexample.

   (a) Every Lebesgue measurable set of strictly positive Lebesgue measure contains a non-empty open interval.

   (b) If f is an integrable function on a measure space (X, S, μ), then $\{x : |f(x)| \neq 0\}$ is a countable union of sets of finite measure.

   (c) The sequence $\{e^{inx} : n = 0, 1, 2, \cdots\}$ is complete in $L^2[-\pi, \pi]$.

McGILL UNIVERSITY

FACULTY OF SCIENCE

FINAL EXAMINATION

MATHEMATICS 189-355B

ANALYSIS IV

Examiner: Professor J.R. Choksi
Associate Examiner: Professor I. Klemes

Date: Wednesday, April 28, 1999
Time: 2:00 P.M. - 5:00 P.M.

INSTRUCTIONS

Calculators not allowed.
Attempt any 6 (SIX) questions.
All questions carry equal marks.

This exam comprises the cover and 2 pages of questions.