McGILL UNIVERSITY

FACULTY OF SCIENCE

FINAL EXAMINATION

MATHEMATICS 189-355B

ANALYSIS II (PART II)

Examiner: Professor V. Havin
Associate Examiner: Professor J.R. Choksi

Date: Friday, April 19, 1996
Time: 2:00 - 5:00 p.m.

Instructions: Solve all problems

This examination paper comprises this cover and 2 pages of questions

Mathematics Final Exam 189-355B Friday, April 19, 1996

Marks:

(10) 1. Suppose ν is a shift invariant measure , dom$\nu = \mathcal{A}_d$ (= the Lebesgue σ-algebra in $\mathbf{R}^d$), $\nu(\underbrace{[0,1)\times\cdots\times[0,1)}_{d}) = 1$. Prove that $\nu = m_d$ (= Lebesgue measure in $\mathbf{R}^d$).

(10) 2. Suppose $A \in \mathcal{A}_1$, $A \subset [0,1]$, $0 < y < m_1(A)$. Prove that there is a Lebesgue measurable $B \subset A$ such that $m_1(B) = y$. (Consider the function $x \mapsto m_1(A \cap [0,x])$, $0 \le x \le 1$).

(10) 3. Suppose E_1, E_2, E_3 are Lebesgue measurable subsets of $[0,1]$, and any $x \in [0,1]$ belongs to at least two of E_j. Prove that one of E_j satisfies the inequality $m_1(E_j) \geq \frac{2}{3}$. (Look at χ_{E_j}).

(20) 4. Let f be a function continuous on $[a,b]$ and differentiable at any point of (a,b). Prove that if f' is bounded, then $f(b) - f(a) = \int\limits_a^b f' dm_1$.

(10) 5. Let $(X, \mathcal{A}, \mu)$ be a measure space, $\mu(X) < +\infty, (f_n)_{n=1}^{\infty}$ a sequence in $L^0(X,\mu)$ (= a.e. finite measurable functions), $f \in L^0(X,\mu)$. Prove that the following are equivalent:

$(A) f_n \xrightarrow{\mu} f$: (B) any subsequence $(f_{n_k})_{k=1}^{\infty} (n_1 < n_2 < \cdots)$ contains a subsubsequence $(f_{n_{k_\ell}})_{\ell=1}^{\infty} (k_1 < k_2 < \cdots)$ such that $f_{n_{k_\ell}} \to f$ a.e.

(10) 6. Suppose $\mu(X) < +\infty$, $p \geq 1$, $(f_n)_{n=1}^{\infty}$ is a sequence in $L^p(X,\mu)$, $\sum\limits_{n=1}^{\infty} \| f_n \|_p < +\infty$.

Prove that $\sum\limits_{n=1}^{\infty} f_n(x)$ absolutely converges a.e., $\sum\limits_{n=1}^{\infty} f_n \in L^p(X,\mu)$, and

$\| \sum\limits_{n=1}^{\infty} f_n \|_p \leq \sum\limits_{n=1}^{\infty} \| f_n \|_p$.

(10) 7. Let $(f_n)_{k=1}^{\infty}$ be a sequence of non-negative functions, $f_n \in L^1(X,\mu)(n = 1,2,\cdots)$, $f \in L^1(X,\mu)$. Prove that if $f_n \underset{n\to\infty}{\longrightarrow} f$ a.e. and $\int_X f_n d\mu \underset{n\to\infty}{\longrightarrow} \int_X f d\mu$, then $\| f_n - f \|_1 \underset{n\to\infty}{\longrightarrow} 0$. (Hint: $(f - f_n)_+ \leq f$). Is it true if we drop the non-negativity assumption?

(10) 8. Suppose $f, g \in L^1(X,\mu)$. Put $F(x,y) := f(x)g(y)(x, y \in X)$. Prove that $F \in L^1(X \times X, \mu \otimes \mu)$, and

$$\int_{X\times X} F d(\mu \otimes \mu) = \int_X f d\mu \cdot \int_X g d\mu.$$

(10) 9. Suppose $f_n \in L^1(X,\mu)(n = 1,\cdots)$, $\sup_n |f_n| \in L^1(X,\mu)$. Prove that the following are equivalent:

(A) $f_n \underset{n\to\infty}{\overset{\mu}{\longrightarrow}} 0$; (B) $\| f_n \|_1 \underset{n\to\infty}{\longrightarrow} 0$.