FACULTY OF SCIENCE

FINAL EXAMINATION

MATHEMATICS 189-355B

ANALYSIS IV

Examiner: Professor I. Klemes
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Date: Monday, 23 April, 2001
Time: 2:00 pm - 5:00 pm

INSTRUCTIONS

This is a closed book examination.
Answer all 6 questions.
Each question is worth 10 marks.
Keep this exam paper.

This exam comprises the cover and 2 pages of questions.

1. (a) Fix a Lebesgue measurable set $E \subset \mathbb{R}$ with $m(E) < \infty$ and let $\epsilon > 0$. Prove that for some n (which may depend on ϵ) we can find bounded intervals $I_1, \ldots, I_n$ such that the set $J := I_1 \cup \ldots \cup I_n$ satisfies $m(E \Delta J) \leq \epsilon$. Here, $E \Delta J$ denotes the symmetric difference.

   (b) Show that in (a) if we impose the additional requirement that $E \subset J$, then the set J may not exist. (Give a counterexample with specific E, ϵ.)

2. (a) State and prove Fatou's Lemma.

   (b) Prove or disprove: If $\{f_n\}$ is a sequence of nonnegative measurable functions in a measure space $(X, \mathcal{M}, \mu)$, then

$$\limsup_{n \to \infty} \left(\int f_n d\mu \right) \leq \int (\limsup_{n \to \infty} f_n) d\mu.$$

3. Evaluate, justifying all limit operations:

$$\lim_{n \to \infty} \int_{\frac{1}{n}}^{2} \frac{ne^x}{n^2 x^2 + \cos^2 x} dx.$$

4. (a) Suppose that $f \in L^+(X, \mathcal{M}, \mu)$ and $\int f d\mu < \infty$. Prove that $f(x)$ is finite for μ–almost all $x \in X$.

   (b) Suppose $\{f_n\} \subset L^2(X, \mathcal{M}, \mu)$ and that for all $N \in \mathbb{N}$,

$$\sum_{k=1}^{N} ||f_{k+1} - f_k||_2 \leq 1.$$

   Prove that the real series $\sum_{k=1}^{\infty} f_k(x)$ converges to a finite limit for μ-almost all $x \in X$.

5. Let $f, f_n \in L^1(X, \mathcal{M}, \mu), n = 1, 2, \ldots$ and suppose that $f_n \to f$ in the metric of L^1. Let $g_k : X \to [-1, 1], k = 1, 2, \ldots$ be measurable functions such that for each fixed $n \in \mathbb{N}$,

$$\lim_{k\to\infty} \int f_n g_k d\mu = 0.$$

Prove that

$$\lim_{k\to\infty} \int f g_k d\mu = 0.$$

6. (a) Let $\phi_1, \phi_2, \ldots$ be an orthonormal sequence of elements of an inner product space and f an element of the space. State the Bessel inequality.

   (b) If $f \in L^2([0, 2\pi])$, prove that

$$\lim_{n\to\infty} \int_0^{2\pi} f(x) \cos nx \, dx = 0.$$

   (c) Let $\delta > 0$ and let $E \subset [0, 2\pi]$ with $m(E) > 0$. Prove that there can be at most finitely many distinct integers n with the property

$$\cos nx \geq \delta \text{ for all } x \in E.$$