FACULTY OF SCIENCE

FINAL EXAMINATION

MATHEMATICS 189-355B

ANALYSIS IV

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Date: Tuesday, April 18, 2000
Time: 2:00 pm - 5:00 pm

INSTRUCTIONS

This is a closed book examination.
Answer all 6 questions.
Each question is worth 10 marks.
Keep this exam paper.

This exam comprises the cover and 2 pages of questions.

1. Let $f : \mathbb{R} \to \mathbb{R}$ be $\mathcal{B}(\mathbb{R}) \to \mathcal{B}(\mathbb{R})$ measurable and let $E = \{(x, y) \in \mathbb{R}^2 : y \leq f(x)\}$. Show that $E \in \mathcal{B}(\mathbb{R}^2)$. You may assume that

$$\mathcal{B}(\mathbb{R}^2) = \sigma(\{A \times B : A, B \in \mathcal{B}(\mathbb{R})\}) = \sigma(\{U : U \text{ is an open subset of } \mathbb{R}^2\}).$$

2. Let $(X, \mathcal{M}, \mu)$ be a measure space and let $f \in L^+$.(This means that $f : X \to [0, \infty]$ and f is measurable.)
   (a) Define $\int f d\mu$.
   (b) If $\int f d\mu < \infty$ show that $f(x) \in \mathbb{R}$ (i.e. $f(x)$ is finite) for almost all $x \in X$.
   (c) If $\epsilon > 0$ and $\int f d\mu < \infty$ show that there exists $E \in \mathcal{M}$ such that $\int_E f d\mu \geq \int f d\mu - \epsilon$.
   (d) If $\int f d\mu = 0$ show that $f(x) = 0$ for almost all $x \in X$.

3. (a) State Fatou's Lemma.
   (b) If $f, f_n \in L^2, ||f_n||_2 = 1,\ n = 1, 2, ...$ and if $f_n(x) \to f(x)$ for all $x \in X$, prove that $||f||_2 \leq 1$.
   (c) In (b), give an example such that $||f||_2 < 1$.

4. (a) State a version of the Dominated Convergence Theorem.
   (b) Evaluate the following limits and justify your work.
      i.
$$\lim_{n\to\infty} \int_1^\infty \frac{ne^{-x}}{n + x} dx.$$
      ii.
$$\lim_{n\to\infty} \int_0^2 \frac{ne^{-x}}{1 + n^2x^2} dx.$$

5. Let $(X, \mathcal{M}, \mu)$ be any measure space. Prove that the metric space L^2 is complete.

6. (a) Consider the functions $f_n(x) = \cos(nx), x \in [0, 2\pi], n = 1, 2, ...$ as points in the metric space $L^2([0, 2\pi])$. Show that the set $\{f_n : n \in \mathbb{N}\}$ is closed and bounded, but not compact.

   (b) If $f : [0,1] \to [0,2]$ and if $L(f)$ and $U(f)$ are the Riemann lower and upper integrals of f, show that there exist measurable functions $\alpha, \beta : [0,1] \to \mathbb{R}$ such that $0 \leq \alpha \leq f \leq \beta \leq 2$ and

$$\int \alpha dm = L(f), \quad \int \beta dm = U(f).$$