1. If A and B are two subsets of $\mathbb{R}^n$, define

$$A + B = \{a + b : a \in A,\ b \in B\}.$$

Prove the following

(a) If A is open B arbitrary, then $A + B$ is open.

(b) If A and B are both compact, then $A + B$ is compact.

(c) If A is compact, B is closed, then $A + B$ is closed.

2. Let (X, ρ) be a metric space, $\{f_n\}$ a sequence of continuous, real valued functions on X.

(a) If f_n converges uniformly on X to a function f, show that f is continuous.

(b) If further $x_n \to x$ in X, show that $f_n(x_n) \to f(x)$.

3. (a) If X is a connected metric space, and f is a continuous function from X to a metric space Y, show that $f(X)$ is connected.

(b) Let f be a continuous function mapping the closed unit interval $[0, 1]$ into itself. Prove that $f(x) = x$ for at least one $x \in [0, 1]$.

4. Let $\{f_n\}$ be a sequence of differentiable, real-valued functions on $[0, 1]$, and suppose there exists $M > 0$ such that

$$|f_n(x)| \leq M, \qquad |f_n'(x)| \leq M, \qquad \forall\ n \in \mathbb{N},\ x \in [0, 1].$$

Show that $\{f_n\}$ has a uniformly convergent subsequence.

5. State the Stone-Weierstrass Theorem for $\mathcal{C}_{\mathbb{R}}(X)$, X a compact metric space.

Let $\mathcal{C}_0$ be the (closed) subspace of $\mathcal{C}([0, 2\pi])$ consisting of continuous functions f such that $f(0) = f(2\pi)$. Show that $\mathcal{C}_0$ can be identified in a natural way with the space $\mathcal{C}_{\mathbb{R}}(\mathbf{T})$ where $\mathbf{T}$ is the unit circle centre the origin in $\mathbb{R}^2$ (i.e., the set

$\{(x, y) : x^2 + y^2 = 1\}$). Hence show that if

$$\mathcal{T} = \left\{ a_0 + \sum_{j=1}^{n} a_j \cos jt + b_j \sin jt :\ a_0, a_j, b_j \in \mathbb{R},\ n \in \mathbb{N}, \quad 0 \leq t \leq 2\pi \right\},$$

then $\mathcal{T}$ is uniformly dense in $\mathcal{C}_0$, i.e. every function in $\mathcal{C}_0$ is the uniform limit of a sequence of functions in $\mathcal{T}$.

6. (a) State the inverse and implicit function theorems.

   (b) Let f be a $\mathcal{C}^1$ function $\mathbb{R} \to \mathbb{R}$, and for $(x, y) \in \mathbb{R}^2$, let $u = f(x),\ v = -y + xf(x)$. If $f'(x_0) \neq 0$ for some point $x_0 \in \mathbb{R}$, show that the map $g(x, y) = (u, v)$ is invertible near (x_0, y) for all $y \in \mathbb{R}$, and the inverse is given by $x = f^{-1}(u),\ y = -v + uf^{-1}(u)$. [State carefully what results you use, including results about functions of one variable.]

   (c) Is it possible to solve the equations

$$\begin{aligned} xy^2 + \ xzu + \ \ yv^2 &= \ 3 \\ u^3yz + \ 2xv - \ u^2v^2 &= \ 2 \end{aligned}$$

   for $u = u(x, y, z),\ v = v(x, y, z)$ near $(x, y, z) = (1, 1, 1),\ (u, v) = (1, 1)$? Compute $\dfrac{\partial v}{\partial y}$ at $(1, 1, 1)$.

7. Prove or disprove any 2 (two) of the following.

   (a) A path connected metric space is connected.

   (b) The space ℓ^∞ is separable.

   (c) In a metric space two disjoint closed sets are contained in disjoint open sets.

McGILL UNIVERSITY

FACULTY OF SCIENCE

FINAL EXAMINATION

MATHEMATICS 189-354A

ANALYSIS III

Examiner: Professor J.R. Choksi
Associate Examiner: Professor S.W. Drury

Date: Monday, December 7, 1998
Time: 2:45 P.M. - 5:45 P.M.

INSTRUCTIONS

NO CALCULATORS ARE PERMITTED.
All questions carry equal marks.
Attempt any 6 (SIX) questions.

This exam comprises the cover and 2 pages of questions.