1. Carefully state the following theorems.
   (i) (5 marks) The Baire Category Theorem.
   (ii) (5 marks) The Stone–Weierstrass Theorem.
   (iii) (5 marks) The Implicit Function Theorem.
   (iv) (5 marks) The Picard Existence Theorem.

2. (i) (5 marks) Write down the definition of the term *metric space*.
   (ii) (5 marks) Write down the definition of the term *open subset*.
   (iii) (5 marks) Define the *geodesic distance* on the unit sphere

$$S^{d-1} = \{x; x \in \mathbb{R}^d, \|x\| = 1\}$$

   in d-dimensional Euclidean space $\mathbb{R}^d$ and show that it is a metric.
   (iv) (5 marks) Show that the geodesic distance metric and the standard metric given by

$$d_{\text{standard}}(x, y) = \|x - y\|$$

   define the same open subsets of S^{d-1}.

3. (i) (5 marks) Write down the definition of the term *complete* as it relates to metric spaces.
   (ii) (5 marks) State the *Contraction Mapping Theorem*.
   (iii) (10 marks) Show that there is a unique continuous function $f : [0,1] \longrightarrow [0,1]$ such that

$$f(x) = \frac{1}{4}\Big(2x + (f(x))^2 + f(\frac{x}{2})\Big).$$

4. (i) (5 marks) How is the product space $X \times Y$ of two metric spaces X and Y defined?
   (ii) (5 marks) Stating carefully any theorem that you may need, show that the product space $X \times Y$ is compact whenever X and Y are.
   (iii) (5 marks) Let A and B be closed bounded subsets of $\mathbb{R}$. Show that the set $\{a + b | a \in A, b \in B\}$ is again closed.
   (iv) (5 marks) Show that the set $\{a + b | a \in A, b \in B\}$ need not be closed if A and B are assumed closed, but not necessarily bounded.

5. (i) (5 marks) What is meant by a *connected* metric space.
   (ii) (5 marks) Define the concept of *component* of a metric space.
   (iii) (5 marks) Give an example a metric space X such that two of its distinct components X_1 and X_2 lie on the same side of every splitting.
   (iv) (5 marks) Let X be a subset of the real line $\mathbb{R}$ with the restriction metric. Suppose that X_1 and X_2 are distinct components of X. Show that there is a splitting of X for which X_1 and X_2 lie on different sides.

6. (i) (5 marks) State carefully the Implicit Function Theorem.
   (ii) (5 marks) State carefully the Parametrization Theorem.
   (iii) (10 marks) Show that the set of points $(x, y, z) \in \mathbb{R}^3$ satisfying the equation $(x + y + z)^3 + 4x^2y^2z^2 = 5$ is an infinitely differentiable surface.

7. (i) (15 marks) Let $f : \mathbb{R}^d \longrightarrow \mathbb{R}$ be a twice continuously differentiable function that has a local minimum point at the origin $\mathbf{0}$. Show that

$$\frac{\partial f}{\partial x_j}(\mathbf{0}) = 0$$

for $j = 1, \ldots, d$ and that

$$\left(\frac{\partial^2 f}{\partial x_j \partial x_k}(\mathbf{0})\right)_{j,k}$$

is a positive semidefinite $d \times d$ matrix.
   (ii) (5 marks) State without proof an approximate converse to the result you have proved in (i).

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FACULTY OF SCIENCE

FINAL EXAMINATION

MATHEMATICS 189-354A

Analysis II (part I)

Examiner: Professor S. W. Drury — Date: Wednesday, December 11, 1996
Associate Examiner: Professor K. N. GowriSankaran — Time: 2:00 P.M. – 5:00 P.M.

INSTRUCTIONS

All seven questions should be attempted for full credit.

This is a closed book examination.
Write your answers in the booklets provided.
All questions are of equal weight, each is alloted 20 marks.

This exam comprises the cover and 2 pages of questions.