1. (a) Determine the domain of analyticity of $f(z) = \text{Log}\ i(z-1)$.
   (b) Make suitable branch cuts and define a branch $f(z)$ of $(z^2+1)^{1/2}$ that is defined on the real axis and such that $f(0) = -1$. Find $f(-1)$. Justify your answer.

2. Let $f(z)$ be a complex function and $z_0 \in \mathbb{C}$.
   (a) Define the following concepts:
      i. z_0 is an isolated singularity of f.
      ii. z_0 is a removable singularity of f.
      iii. z_0 is a pole of f.
      iv. z_0 is an essential isolated singularity of f.
   (b) Explain how to extend the definitions in (a) to the case $z_0 = \infty$.

3. For the following functions $f(z)$ determine the type of singularity at the point z_0 indicated.
   (a) $f(z) = e^{-1/z}\sin z^2;\ z_0 = 0.$
   (b) $f(z) = \dfrac{1+z}{1-z};\ z_0 = \infty.$
   (c) $f(z) = \text{Log}\dfrac{z-1}{z+1};\ z_0 = 1.$
   (d) $f(z) = \dfrac{1}{1-\cos z};\ z_0 = 0.$

4. For the functions $f(z)$ and points z_0 of 3), determine whether $f(z)$ has a Laurent expansion at z_0. Where possible, find the order and the residue of f at z_0.

5. (a) Suppose $f(z)$ has an isolated singularity at $z_0 \in \mathbb{C}$. Show that the following are the same:
      i. the coefficient of $(z-z_0)^{-1}$ in the Laurent expansion of $f(z)$ at z_0.
      ii. $\dfrac{1}{2\pi i}\displaystyle\int_{|z-z_0|=\varepsilon} f(z)dz$ (for $0 < \varepsilon$ sufficiently small).
   (b) Suppose that $f(z)$ has a pole or removable singularity at z_0. Let $g(z) = \dfrac{f'(z)}{f(z)}$. Show that $\text{Res}(g(z); z_0)$ is defined and equals the order of $f(z)$ at z_0.

6. Determine the following integrals. Use residues and contour integration where appropriate. Justify your steps.
   (a) $\displaystyle\int_{|z|=2} \frac{1}{z^2+z+1}\,dz;$ (b) $\displaystyle\int_0^{2\pi} \frac{\sin\theta}{2+\cos\theta}\,d\theta;$ (c) $\displaystyle\int_0^{\infty} \frac{\cos x}{x^2+1}\,dx.$

McGILL UNIVERSITY

FACULTY OF SCIENCE

FINAL EXAMINATION

MATHEMATICS 189-316A

FUNCTIONS OF A COMPLEX VARIABLE

Examiner: Professor K.P. Russell
Associate Examiner: Professor J.C. Taylor

Date: Thursday, December 10, 1998
Time: 2:00 P.M. - 5:00 P.M.

This exam comprises the cover and 1 page of questions.