McGILL UNIVERSITY
FACULTY OF SCIENCE
FINAL EXAMINATION

MATHEMATICS 189-315A
ORDINARY DIFFERENTIAL EQUATIONS I

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Date: Friday, December 15, 1995
Time: 14:00 hrs - 17:00 hrs

Instructions: NO BOOKS OR CALCULATORS

This exam comprises the cover, 2 pages of questions and 1 page containing a table of Laplace transforms.

1. (a) Find the inverse Laplace transform of

$$F(s) = \frac{e^s}{s^2} + \frac{s+4}{(s^2+1)(s-1)}.$$

(b) Solve the initial value problem

$$y'' + y = g(t) + \delta(t-2)$$

$$y(0) = 1, \qquad y'(0) = 0$$

where

$$g(t) = \begin{cases} 0, t \leq 0; \\ t, 0 < t \leq 1; \\ 1, t > 1 \end{cases}$$

and $\delta(t)$ is the Dirac delta function.

2. Consider the equation

$$(x-2)(x+3)y'' + 2(x+1)y' + 3y = 0$$

(a) Find the singular points.

(b) Find the recurrence relation satisfied by the coefficients a_n for solutions of this equation of the form $y = \sum_{n\geq 0} a_n x^n$.

(c) Find the first 4 terms $a_0 + a_1 x + a_2 x^2 + a_3 x^3$ in the Taylor series expansion of each of two linearly independent solutions $y = \sum_{n\geq 0} a_n x^n$ centered about $x = 0$.

3. (a) When $x > 0$, find the general solution to the equation

$$x^2 y'' + xy' - 4y = e^{-x^2}.$$

(b) Suppose $x > 0$. One solution of the equation

$$x^2 y'' - (x - \frac{3}{16})y = 0$$

is

$$y_1(x) = x^{1/4} e^{2\sqrt{x}}.$$

Find a second linearly independent solution y_2 using the method of reduction of order.

4. (a) Show that for any real number α, $x = 1$ is a regular singular point of the equation

$$(1 - x^2)y'' - 2xy' + \alpha(\alpha + 1)y = 0$$

(b) Suppose $x > 1$. Determine the indicial equation, its roots r, and the recurrence relation satisfied by the a_n, for one solution of the equation (as a series $y = (x-1)^r \sum_{n \geq 0} a_n (x-1)^n$). (HINT: Substitute $t = x - 1$ and find a solution as a function of t.)

(c) Show that if $\alpha = N$ or $\alpha = -N - 1$ for $N = 0, 1, 2, ...$ then the solution found in part (a) is a polynomial in $x - 1$.

(d) If $\alpha = N$ for $N = 0, 1, 2, \ldots$, determine the degree of this polynomial in terms of N.

5. (a) Use the improved Euler method with a step size $h = 0.1$ to compute an approximate value at $x = 0.2$ for the solution to the initial value problem

$$y' = x + 3y, \qquad y(0) = 2$$

(b) Solve the initial value problem

$$(2x^2 \sin y + xye^x)dx + (x^3 \cos y + xe^x + 3xy^2)dy = 0$$

$$y(1) = -2.$$