1. For each of the given subsets W of the given vector space V over the given field F, determine whether or not W is a subspace of V. Justify your answers.

   (a) $V = \mathbb{R}^{\mathbb{R}}$, $F = \mathbb{R}$, $W = \{f \in V \mid f'' \text{ exists and } xf''(x) + f(x) = 0 \text{ for all } x \in \mathbb{R}\}$;

   (b) $V = \mathbb{R}^{2\times 2}$, $F = \mathbb{R}$, $W = \{X \in V \mid X^2 = 0\}$;

   (c) $V = \mathbb{C}^{\infty}$, $F = \mathbb{C}$, $W = \{x \in V \mid x_{n+2} + ix_{n+1} + (n+i)x_n = 0 \text{ for all } n \geq 0\}$;

   (d) $V = \mathbb{C}^{2\times 2}$, $F = \mathbb{C}$, $W = \{X \in V \mid \overline{X}^T = X\}$.

2. Determine whether or not the given sequences of vectors in the given vector space V over the given field F are linearly independent. Justify your answers.

   (a) $V = \mathbb{R}^{\mathbb{R}}$, $F = \mathbb{R}$, $f_1(x) = e^{2x}, f_2(x) = e^{3x}, f_3(x) = e^{5x}$;

   (b) $V = \mathbb{C}^{[0,1]}$, $F = \mathbb{C}$, $f_1(x) = e^{ix}, f_2(x) = \cos(x), f_3(x) = \sin(x)$;

   (c) $V = \mathbb{C}^{[0,1]}$, $F = \mathbb{C}$, $f_1(x) = (x+i)^2, f_2(x) = (x-i)^2, f_3(x) = (x+1)^2, f_4(x) = (x-1)^2$;

   (d) $V = \mathbb{R}^{\mathbb{R}}$, $F = \mathbb{R}$, $f_1(x) = e^x, f_2(x) = xe^x, f_3(x) = \sin(x)$.

3. Let V be the vector space of infinitely differentiable real-valued functions on the real line and let $W = \{f \in V \mid f'' - 7f' + 10f = 0\}$.

   (a) Prove that W is a subspace by finding a polynomial in D, the differentiation operator, whose kernel is W.

   (b) Show how, by factoring the polynomial found in (a), a basis of W can be found.

   (c) Show that $T : W \to \mathbb{R}^2$, defined by $T(f) = (f(0), f'(0))$, is an isomorphism of vector spaces.

4. Find the orthogonal projection of the column vector $Y = [a, b, c, d]^T$ on the column space of the matrix

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -2 & -1 \\ 1 & 1 & -1 \\ 1 & 2 & -1 \end{bmatrix}$$

   and show how this can be used to find the least squares solution of the system of equations $AX = Y$.

5. Let V be the real vector space of polynomials $p(t) = a_1 + a_2t + a_3t^2 + a_3t^3$ and let W be the real vector space of polynomials $q(t) = b_1 + b_2t + b_2t^2$. Let $T : V \to W$ be defined by $T(p(t)) = 2t^2p''(t) - 12p(t)$.

   (a) Show that T is linear.

   (b) Find bases for the kernel and image of T.

   (c) Find the matrix of T with respect to the bases $1, t, t^2, t^3$ of V and $1, t, t^2$ of W.

6. The state of a discrete dynamical system after n intervals of time is described by the system of equations

$$\begin{aligned} x_{n+1} &= .4x_n + .5y_n \\ y_{n+1} &= -.315x_n + 1.2y_n. \end{aligned}$$

   (a) Find the general solution.

   (b) Show that x_n, y_n tend to 0 as n goes to infinity for any choice of initial values x_0, y_0.

7. Let $A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$.

   (a) Given that, $(A - I)^2(A - 4) = 0$, find an orthogonal matrix P such that $P^{-1}AP$ is a diagonal matrix.

   (b) Using (a), find the general solution of $\frac{dX}{dt} = AX$.

   (c) Using (a), evaluate the triple integral

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-2(x^2+y^2+z^2+xy+xz+yz)}\,dxdydz.$$

   Recall that $\int_{-\infty}^{\infty} e^{-t^2}dt = \sqrt{\pi}$.

McGILL UNIVERSITY

FACULTY OF ENGINEERING

FINAL EXAMINATION

MATHEMATICS 189-270A

APPLIED LINEAR ALGEBRA

Examiner: Professor J. Labute
Associate Examiner: Professor D. Sussman

Date: Monday, December 13, 1999
Time: 2:00 P.M. - 5:00 P.M.

INSTRUCTIONS

Attempt all Questions.
Explain Clearly and Justify All Your Work.
All Questions Are Of Equal Value.
University Standard Calculators Allowed.

This examination consists of two pages of questions plus the cover page.