April 2, 1998

FINAL EXAMINATION

189-266B: Linear Algebra and Boundary Value Problems

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(18)
1. Consider the diffusion equation with a non-constant source term

$$u_t = u_{xx} + e^{-t} \quad , \quad 0 < x < 1 \ , \quad t > 0$$
$$u(0,t) = u_0 \ , \quad u(1,t) = u_1 \ , \quad t > 0$$
$$u(x,0) = 0 \ , \quad 0 < x < 1$$

Find
(a) the steady state solution $u_s(x)$,
(b) the transient solution $w(x,t)$ in the form $w(x,t) = \Sigma_{n=1}^{\infty} a_n(t) \sin n\pi x$,
(c) the complete solution $u(x,t)$.

(12)
2. Find the solution $u(r,\theta)$ of Laplace's equation $\frac{\partial^2 u}{\partial^2 r} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial^2 \theta} = 0$, in the semi-circular region $r < 1$, $0 < \theta < \pi$, satisfying the boundary conditions

$$u(r,0) = 0, \ \ u(r,\pi) = 0, \quad 0 \le r < 1,$$
$$u(\pi,\theta) = \sin 3\theta, \quad 0 \le \theta \le \pi.$$

Assume that u is single valued and bounded.

(14)
3.(a) Show that the wave equation

$$u_{tt} = a^2 u_{xx}$$

can be reduced to the form $u_{\xi\eta} = 0$ by the change of variables $\xi = x - at$, $\eta = x + at$. Show that $u(x,t)$ must be of the form

$$u(x,t) = F(x - at) + G(x + at),$$

where F and G are arbitrary functions.

3.(b) Using (a) or other method solve

$$u_{tt} = a^2 u_{xx} \ , \quad 0 < x < 1 \ , \ t > 0$$
$$u(0,t) = u(1,t) = 0 \ , \quad t > 0$$
$$u(x,0) = \sin \pi x \ , \quad u_t(x,0) = 0 \ , \quad 0 < x < 1$$

(16)
4. Let A be the matrix

$$A = \begin{pmatrix} 4 & 2 & 2 \\ 2 & 4 & 2 \\ 2 & 2 & 4 \end{pmatrix}$$

(a) Find the characteristic equation, eigenvalues, and eigenvectors of A (Hint: 2 is a root of the characteristic equation.)

(b) Find a matrix P that orthogonally diagonalizes A, and determine $P^{-1}AP$.

(c) Find the general solution $x(t)$ of the system $\mathbf{x}' = A\mathbf{x} + \mathbf{c}$, where $\mathbf{x} = (x_1, x_2, x_3)^T$ and $\mathbf{c} = (1, 0, 0)^T$.

(16)
5. Consider the following mass-spring system.

Assuming that the only forces acting on m_1 and m_2 are F_1 and F_2 respectively, the differential equations governing the displacements x_1 and x_2 of the two masses are

$$\begin{aligned} m_1 x_1'' &= -k_1 x_1 - k_{12}(x_1 - x_2) + F_1 \\ m_2 x_2'' &= -k_{12}(x_2 - x_1) - k_2 x_2 + F_2. \end{aligned}$$

(a) Taking $m_1 = m_2 = k_1 = k_{12} = k_2 = 1$, and $F_1 = F_2 = 0$, find the general solution using the diagonalization method.

(b) Let $m_1 = m_2 = k_1 = k_{12} = k_2 = 1$, and $F_1 = \sin 2t$ and $F_2 = 0$, find a particular solution in the form

$$\begin{aligned} x_1(t) &= \xi_1 \sin 2t \\ x_2(t) &= \xi_2 \sin 2t \end{aligned}$$

(12)
6.(a) For

$$A = \begin{pmatrix} 2 & -5 \\ 1 & -2 \end{pmatrix}$$

find e^{At}.

6.(b) Find the solution of the initial value problem

$$\begin{aligned} x_1' &= 2x_1 - 5x_2 \\ x_2' &= x_1 - 2x_2 \\ x_1(0) = 1, &\quad x_2(0) = 0. \end{aligned}$$

(12)
7.(a) Express the quadratic form $5x^2 - 4xy + 8y^2$ in the matrix notation $x^T Ax$, where A is a symmetric matrix.

7.(b) Find the maximum and the minimum values of the quadratic form in (a) subject to the contraint $x^2 + y^2 = 1$, and determine the values of x and y at which the maximum and minimum occur.

7.(c) Rotate the coordinate axes to put the conic $5x^2 - 4xy + 8y^2 - 36 = 0$ in a standard position. Name the conic and give its equation in the final coordinate system.