1. (a) (7 marks) Solve

$$\frac{\partial \psi}{\partial t} = \frac{\partial^2 \psi}{\partial x^2} + 3; \quad 0 < x < \pi, \; t > 0$$

(i) $\psi_x(0,t) = 0$, (ii) $\psi_x(\pi,t) = 0$, (iii) $\psi(x,0) = 5$.

(b) (12 marks) Solve

$$\frac{\partial \psi}{\partial t} = \frac{\partial^2 \psi}{\partial x^2} + h(x,t); \quad 0 < x < \pi, \; t > 0$$

(i) $\psi(0,t) = F(t)$, (ii) $\psi(\pi,t) = G(t)$, (iii) $\psi(x,0) = f(x)$.

2. (a) (9 marks) Obtain the general solution of Laplace's equation in spherical coordinates with no θ dependence, i.e., $\nabla^2\psi(r,\varphi) = 0$, $0 \le \varphi \le \pi$, with ψ finite at $\varphi = 0$ and $\varphi = \pi$. Explain carefully all your steps.

(b) (7 marks) Solve

$$\nabla^2\psi(r,\varphi) = 0, \quad r > a, \; 0 \le \varphi \le \pi$$

(i) $\left[\frac{\partial \psi}{\partial r}\right]_{r=a} = 0$, (ii) $\lim_{r\to\infty}[\psi(r,\varphi) - V_0 r\cos\theta] = 0$ and interpret physically.

3. (14 marks) Solve

$$\nabla^2\psi(r,\theta) = 0; \quad 1 < r < e, \; 0 < \theta < \alpha$$

(i) $\psi_r(1,\theta) = 0$, (ii) $\psi_r(e,\theta) = 0$, (iii) $\psi(r,0) = 0$, (iv) $\psi(r,\alpha) = f(r)$.

Leave your answer in SIMPLEST form. Note $e = 2.718.....$

Hint: Find the steady-state temperature distribution in the shaded region.

4. (12 marks) Solve

$$\nabla^2\psi(r,z) = -q(r,z); \quad 0 \le r < b, \; 0 < z < \pi$$

(i) $\psi(r,0) = 0$, (ii) $\psi(r,\pi) = 0$, (iii) $\psi(b,z) = 0$.

5. (a) (8 marks) Give a necessary and sufficient condition for the integral

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} \exp\left[-\sum_{j=1}^{N}\sum_{i=1}^{N} x_i T_{ij} x_j\right] dx_1 dx_2 \cdots dx_N$$

to converge and derive carefully the simplest value when convergent.

Hint: You may use $\int_0^\infty e^{-x^2} dx = \sqrt{\pi}/2$.

(b) (4 marks) Does

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-[2x^2+4y^2+6z^2-4xy+4xz]}dxdydz$$

converge? If so, evaluate the integral.

(c) (4 marks) Given that

$$f(x_1,x_2,x_3) = 35 - 6x_1 + 2x_3 + x_1^2 - 2x_1x_2 + 2x_2^2 + 2x_2x_3 + 3x_3^2$$

has an extremum at $P(8,5,-2)$. Find the nature of that extremum.

6. (a) (7 marks) Solve the system of differential equations

$$\begin{aligned} \dot{x}_1 &= 2x_1 - 2x_2; \qquad & x_1(\pi/2) = -2 \\ \dot{x}_2 &= 4x_1 - 2x_2; \qquad & x_2(\pi/2) = 1 \end{aligned}$$

by using the exponential matrix method.

(b) (4 marks) Obtain the Green's matrix for the system

$$\begin{aligned} \dot{x}_1 &= 2x_1 - x_2 + f_1(t); \qquad & x_1(\pi/2) = -2 \\ \dot{x}_2 &= 4x_1 - x_2 + f_2(t); \qquad & x_2(\pi/2) = 1 \end{aligned}$$

and write the solution in terms of $f_1(t)$ and $f_2(t)$. It is NOT necessary to simplify your answer.

7. (12 marks) Solve

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\dfrac{2}{t^2} & \dfrac{2}{t} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} t^2 \\ 3t \end{bmatrix} \begin{matrix} ;\ x_1(1) = 2 \\ ;\ x_2(1) = 3 \end{matrix}$$

Good Luck!

McGILL UNIVERSITY

FACULTY OF ENGINEERING

FINAL EXAMINATION

MATHEMATICS 189-266A

LINEAR ALGEBRA & BOUNDARY VALUE PROBLEMS

Examiner: Professor C. Roth — Date: Monday, December 20, 1999

Associate Examiner: Professor N.G.F. Sancho — Time: 9:00 A.M. - 1:00 P.M.

This examination consists of the cover page, two pages of questions plus a page of useful information.