1. (a) (8 marks) Obtain the general solution of Laplace's equation in spherical coordinates with no θ dependence, i.e. $\nabla^2\psi(r,\varphi) = 0,\ 0 \le \varphi \le \pi$, with ψ finite at $\varphi = 0$ and $\varphi = \pi$.

   (b) (10 marks) Find the steady-state temperature (potential) distribution inside a hemisphere if the spherical part is maintained at a temperature $f(\cos\varphi)$ and the flat part is insulated against the flow of heat.

   Hints: (1) Solve $\nabla^2\psi(r,\varphi) = 0$

   (i) $\psi(a,\varphi) = f(x)$, (ii) $\left[\dfrac{\partial\psi}{\partial z}\right]_{\varphi=\pi/2} = 0.$

   $x = \cos\varphi$

   (2) Show that the condition of insulation on the flat face, i.e.

$$\left[\frac{\partial\psi}{\partial z}\right]_{\varphi=\pi/2} = 0 \text{ implies } \left[\frac{\partial\psi}{\partial\varphi}\right]_{\varphi=\pi/2} = 0.$$

   (3) Your solution should involve Legendre polynomials of even order.

2. (15 marks) Find the temperature distribution in a circular plate of radius b if the circumference is maintained at a temperature T_0, the initial temperature is $f(r)$ and there is constant heat generation Q.

   Hint: Solve $\nabla^2\psi(r,t) = -\dfrac{Q}{K};\ 0 \le r \le b,\ t > 0.$

   (i) $\psi(b,t) = T_0$, (ii) $\psi(r,0) = f(r)$.

3. (15 marks) Solve and interpret physically:

$$\frac{\partial\psi}{\partial t} = \frac{\partial^2\psi}{\partial x^2} - 12x;\ 0 < x < 1,\ t > 0.$$

   (i) $\psi(0,t) = 1$, (ii) $\left[\dfrac{\partial\psi}{\partial x}\right]_{x=1} = -3[\psi(1,t) - 9]$,

   (iii) $\psi(x,0) = 2x^3 + 3x - 5$.

   Leave your answer in simplest form.

4. (10 marks) Solve and interpret physically

$$\nabla^2\psi(x,y) = -q(x,y); \quad 0 < x < \pi,\ 0 < y < \pi$$

(i) $\psi(x,0) = 0$, (ii) $\psi(x,\pi) = 0$, $\psi(0,y) = 0$, (iv) $\psi(\pi,y) = 0$.

5. (8 marks) (a) A linear operator τ maps the vector $5\vec{\imath} + 2\vec{\jmath}$ onto $44\vec{\imath} + 20\vec{\jmath}$ and the vector $3\vec{\imath} - 4\vec{\jmath}$ onto $16\vec{\imath} - 14\vec{\jmath}$. Through what angle must the $[\vec{\imath}, \vec{\jmath}]$ basis vectors be rotated in order for the matrix representative of τ to be diagonal?

(b) (5 marks) (i) Use matrix methods to identify and sketch the central conic

$$8x_1^2 + 4x_1x_2 + 5x_2^2 = 36.$$

(ii) Letting y_1, y_2 denote the axes with respect to which the equation has no cross terms, obtain the matrix of transformation from the x_1, x_2 axes to the y_1, y_2 axes.

(iii) Obtain the angle of rotation corresponding to the above transformation.

(c) (4 marks) Obtain the general solution of the system of differential equations

$$\begin{aligned} \dot{x}_1 &= 8x_1 + 2x_2 \\ \dot{x}_2 &= 2x_2 + 5x_1 \end{aligned}$$

by the method of diagonalization.

6. (12 marks) Consider the system of vibrating masses below where x_1 and x_2 are measured from their respective equilibrium positions.

(a) Find the normal frequencies of vibration.

(b) Find the normal modes of vibration.

(c) If the system has initial displacement $X_0 = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$ and initial velocity $\dot{X} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ determine the subsequent motion. Assume that there is no friction in the system.

7. (13 marks)

(a) For $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 2 & 5 \\ 0 & -1 & -2 \end{bmatrix}$ find e^{At}.

(b) Solve the initial value problem:

$$\begin{aligned} \ddot{x} &= 2\dot{x} + 5y \\ \dot{y} &= -\dot{x} - 2y \end{aligned} \qquad \begin{aligned} x(0) &= 1 \\ \dot{x}(0) &= -1 \\ y(0) &= 1. \end{aligned}$$

(c) Obtain the Green's matrix for the system

$$\begin{aligned} \ddot{x} &= 2\dot{x} + 5y + f_1(t) \\ \dot{y} &= -\dot{x} - 2y + f_2(t) \end{aligned} \qquad \begin{aligned} x(0) &= 1 \\ \dot{x}(0) &= -1 \\ y(0) &= 1 \end{aligned}$$

and write the solution in terms of $f_1(t)$ and $f_2(t)$. It is NOT necessary to simplify your answer.

Good Luck!

McGILL UNIVERSITY

FACULTY OF ENGINEERING

FINAL EXAMINATION

MATHEMATICS 189-266A

LINEAR ALGEBRA & BOUNDARY VALUE PROBLEMS

Examiner: Professor C. Roth	Date: Monday, December 14, 1998
Associate Examiner: Professor D. Sussman	Time: 9:00 A.M. - 1:00 P.M.

This exam comprises the cover, 3 pages of questions and 1 page of useful information.